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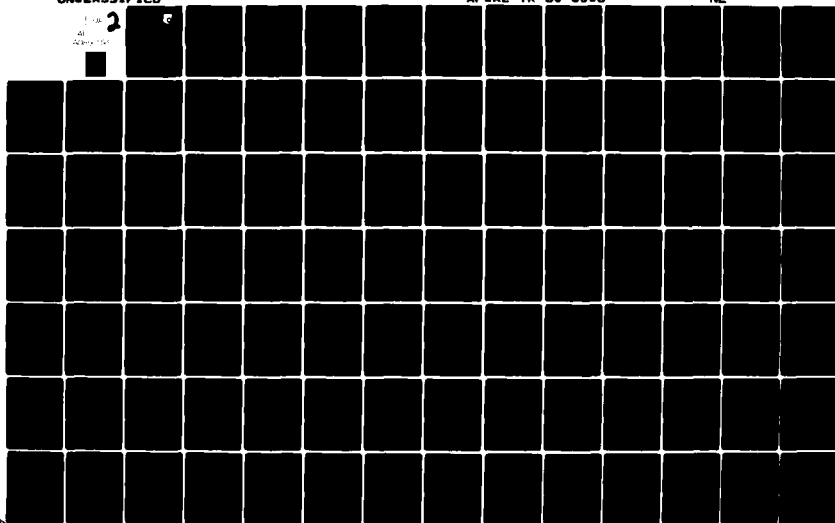
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FORCE METHOD OPTIMIZATION

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*BELL AEROSPACE TEXTRON
BUFFALO, NEW YORK 14240*

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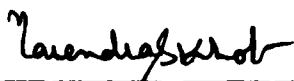
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Structural Optimization Force Method of Finite Element Analysis Minimum Weight Design Stress Constraints			
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<p>→ To overcome problems arising from the rigorous representation of stress constraints in structural optimization, the use of the force method is demonstrated.</p> <p>The force method of finite element analysis is used to transform a stress-constrained redundant structure into a stress and displacement constrained determinate structure for the purposes of optimization. The resulting problem is non-linear, but the use of a linear programming stage is effective in →</p>			

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providing a rapid convergence on the optimum design in many cases. Problems which hitherto required many iterations to converge are now solvable in only one or two steps. The rate of constraint numbers versus numbers of variables is investigated analytically using a 22-bar truss problem as an example. The vital key in optimization technology is identified as the identification of active constraints.

The incorporation of displacement constraints into the optimization program is also presented. Examples of problems solved by this approach are included. Discussion of difficulties encountered in the development work also provides indicators for future areas of research.

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FOREWORD

This report describes the work performed by Bell Aerospace Textron, a Division of Textron, Inc., Buffalo, New York. The work was sponsored by the Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio, under contract F33615-77-C-3032.

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1.0 INTRODUCTION

a. General

Progress in the development of practical structural optimization technology has been consistently characterized by a series of major advances, followed by periods of consolidation and even retrenchment. Design or optimization is fundamentally a more sophisticated concept than structural analysis and it is therefore only to be expected that the derivation and development of new techniques will be a much slower process than, say, the development of finite element analysis methods.

After the major concentration of attention in the 60's on mathematical programming techniques reached what was effectively a computational stalemate, the development of optimality criteria methods⁽¹⁾ in the early 70's appeared to offer major improvements over any other optimization methods currently in vogue. Certain deficiencies in optimality criteria methods were recognized but the computational advantages of some approximate methods were sufficient to warrant the development of large scale computer programs, such as OPTIM^(2,3) and ASOP⁽⁴⁾.

Following the initial rush of development has been a larger period of study, during which the deeper implications and above-mentioned deficiencies in the approximate optimality criteria methods have been evaluated. Although a number of differing approaches to redesign strategies have been proposed by a number of authors, it has been demonstrated⁽⁵⁾ that the vast majority of these approaches are really variations of a common theme. The essential

differences being merely resident in the selection of arbitrary coefficients and other parameters which might appear to influence convergence characteristics.

In spite of this work, it appears that little real headway has been made in solving some of the fundamental problems in optimization. The use of a Lagrangian formulation is well established as a starting point for optimization but there are still many outstanding obstacles between the initial concept and workable optimization tools. For single displacement constraints, there is no problem. For multiple displacement constraints, the governing optimality criteria can be established, but the solution of these nonlinear equations with unknown constraint population is by no means a solved problem. For stress constraints, even the establishment of valid optimality criteria has not hitherto been successfully performed. In summary, it may be stated that there are no adequate operational methods for optimizing structures subject to multiple constraints on strength (stress), stiffness (displacement) and fabrication limits (minimum member sizes). Other types of constraints-stability, dynamic response, flutter and aeroelasticity are of prime importance, but they are generally and generically related to the three basic constraints. Hence, the approach is to solve the problems associated with the basic constraints and the more sophisticated constraints will follow.

For stress constraints, the classic, time-honored and erroneously fully stressed design (FSD) method based upon the stress-ratio technique

is most widely used. FSD, while inaccurate, has the overwhelming merit of being extremely simple and economical to use. For the treatment of displacement constraints no equivalent existed in the 60's and only rigorous (and costly) mathematical programming methods could be used.

The development of the methods now generally labeled as optimality criteria was the result of attempts to fill this need for an approach to displacement constraints. The optimality criteria methods are iterative (as are mathematical programming methods) but the number of iterations required for convergence appeared to be largely independent of the number of variables (the downfall of pure math programming).

A considerable degree of success was achieved, particularly using the envelope method, but the limitations of the approach were recognized. The solution is exact for a single equality (displacement) constraint only. For multiple constraints, the procedure for active/passive partitioning of members is approximate and there is no capability for treating problems involving mixed equality and inequality constraints. In addition, the incorporation of stress and fabrication constraints is effectively based upon the FSD method.

Work has been carried on by a number of research teams in attempts to (a) solve the problem of multiple displacement constraints in a rigorous (or even non-rigorous but practical) manner - but with little real success and (b) improve upon stress-ratio FSD for strength constraints. For the latter, concepts such as strain energy density, adaptive steps, etc., have been investigated but none of the approaches have been totally effective.

The finite element displacement method has dominated the field of analysis for many years and has been used almost exclusively in optimization work. The force method has largely been ignored in analysis and consequently has not been considered of general or particular relevance to optimization. Some recent use has been made of the force method^(6, 7), but this was primarily to reduce the computational effort associated with iterative analysis. At Bell, the role of the force method was examined⁽⁸⁾ to determine the possibility of a fundamental integration of the analysis and redesign philosophy. The preliminary results did indicate a potential for the rigorous incorporation of stress constraints. While extremely simple, the approach of using the force method concept to overcome the difficulties associated with strength optimization of redundant systems appears to be entirely novel. If the internal redundant forces are considered to be self-equilibrating external forces, the structure considered becomes effectively statically determinate. The optimality criteria for such a determinate structure can then be derived rigorously for either displacement or stress constraints. The compatibility conditions, associated with the redundancies, now become zero-valued quasi-external displacement constraints. Additional optimality criteria are derived to permit the consideration of the redundant forces as subsidiary variables.

The problem of stress constraints is thus transformed into one involving displacement constraints. This general concept has also been proposed for bar-type structures⁽⁹⁾, whereby the maximum allowable

stress in a bar element is related to strain and hence the relative displacements at each end of the bar.

While this approach is undoubtedly entirely valid, its extension to other types of finite elements may present a major problem. Also, it is believed that it may require more computational effort than the current force method approach.

b. Force Method Background

With the generation of optimality criteria for both stress and displacement constraints, the next problem is the solution of the resulting set of nonlinear equations. The most appropriate approach is through the use of some form of Newton-Raphson technique.

Attempts to solve the linearized form of the total set of governing equations was relatively unsuccessful - principally due to the inability of the search technique to identify and distinguish between active and inactive constraints.

This difficulty was largely overcome by the introduction of a linear programming technique, which provided an estimate of the optimal population of active constraints. This approach which is based upon a local linearization of the domain, leads to the selection of full vertex corresponding to the satisfaction that the number of constraints is equal to the number of variables.

For stress constrained problems in which this situation applies at the optimum, the use of the linear programming approach has been found to generate this optimal design in only one or two iterative steps. The validity of the optimality is demonstrated by checking that all Lagrangian multipliers are non-negative.

Historically, in test problems involving only stress and minimum size constraints, the optimum designs determined have appeared to be full vertices--i.e., the number of constraints is equal to the number of variables. Even when it has not been possible to determine the optimum design rigorously, there have been indications that the optimum design tends to have a large number of active constraints. The question of number of active constraints is important since it affects significantly the strategy required. Therefore, for a number of variations on a 22-bar truss theme, analytic studies were undertaken to determine the optimal design and demonstrate rigorously the active constraint populations. It was shown that, depending on the values of certain parameters, some optimal designs had 22 constraints and others only 21.

For such problems, in which the optimum does not occur at a full vertex, the linear programming approach can still play a dominant role. The linear programming stage will lead to a full vertex design but one for which some Lagrangian multipliers are negative. These correspond to constraints which should not be active at the optimum design and hence should be discarded. For the reduced set of constraints, the Newton-Raphson method can be used satisfactorily. The key to the solution of the optimal system is a good reliable indication of which constraints are active at the optimum design.

When additional displacement constraints are considered, the basic approach is essentially unaltered. Displacement constraints appear even

in the stress constraint case--as compatibility conditions. Thus the mathematical formulation is readily extendable to represent real external displacement constraints. What is different is the uncertainty regarding which of these displacement constraints is active. By definition, all compatibility conditions are always active constraints; additional displacement constraints need not be active. Also, displacement constraints tend to dominate, sometimes to the exclusion of other types of constraints. Hence, a displacement constrained optimum design may have a much sparser constraint population than a full vertex design.

Nonetheless, the same strategy can be applied as for the purely stress limited cases. The constraint population predicted by the linear-programming appears to be less valid and greater reliance has to be placed on constraint acquisition and discard algorithms which are incorporated in the full Newton-Raphson search.

c. Report Organization

In Section 2 of this report, a full technical discussion is presented of the various stages in the development of a force method optimization program. Section 2.1 redefines the development of optimality criteria for displacement constraints and highlights the problem of incorporating stress constraints in a rigorous manner. The fundamentals of the force method are presented for completeness in Section 2.2, since this analysis concept is generally neglected in modern finite element work. This discussion also includes consideration of structure cutters. Section 2.3 pulls together the optimality criteria approach and the force method and defines the overall

approach and governing equations. The problems and difficulties encountered in the early work are discussed in Section 2.4 leading into the development of the linear programming approach - Section 2.5. The remainder of the section deals with the analytic study of the 22-bar truss problem, the concept of constraint discard and the generalized Newton-Raphson approach, and finally the introduction of discrete displacement constraints. Results of various optimization problems are inserted throughout these discussions where relevant.

Section 3 outlines the development and logic of the computer program which evolved from this study. The final section summarizes the work by presenting conclusions and recommendations.

2.0 TECHNICAL DISCUSSION

2.1 Optimality Criteria

The concept of developing optimality criteria for structural optimization problems using a Lagrangian approach is well established. This was the basis of the envelope method used in the development of the OPTIM series of programs^(2, 3). A number of variations on the optimality criteria solution algorithm have been developed, but a comprehensive study⁽⁵⁾ has effectively demonstrated that all these algorithms are essentially similar. It must be noted, again, that all these approaches have been developed using a vigorous basis for the consideration of displacement (and minimum size) constraints. Stress constraints, where they have been produced, have been treated in a more approximate manner

To understand this, it is appropriate to recapitulate the basic tenets of the Lagrangian approach to the development of optimality criteria.

The general problem is the minimization of the weight W of a structure subject to a number of inequality constraints of the form $C_j - \bar{C}_j \leq 0$ acting on various behavioral response characteristics of the structure.

While many types of responses can be constrained, for the purposes of this study attention is confined solely to displacements, stresses, and minimum member sizes. Other types of constraints -- stability,

dynamic response, flutter, etc., are of prime importance in structural design, but they are generally and generically related to the three basic constraints mentioned above.

The standard approach is to form a Lagrangian

$$L = W + \sum_j \lambda_j (C_j - \bar{C}_j) \quad (1)$$

where W is the merit function (weight) of the system to be minimized.

C_j is the value of some response characteristic which is constrained to be less than or equal to an allowable value \bar{C}_j

and λ_j is an undetermined Lagrangian multiplier.

Differentiation of Equation 1 with respect to the primary design variables will yield a set of optimality criteria which must be satisfied by the optimum design. Differentiation of Equation 1 with respect to the secondary variables (Lagrangian multipliers) yields explicitly the constraint conditions. One major problem in this or any other approach to structural optimization is the determination of which constraints are active (e.g., satisfied as equalities) and which are inactive (inequalities). This problem will be addressed in a later section.

Under the assumption that the constraint population can be determined or defined, solution of the combined set of optimality and constraint equations (usually nonlinear) will indeed yield the optimum design.

To have a Lagrangian, Equation 1, which is capable of explicit differentiation for solution, it is necessary to introduce certain

assumptions and limitations on the mathematical model of the system.

For the merit function, two conditions must be satisfied:

- (1) The total weight is the sum of the component member weights.
- (2) The weight of each member is a linear function of its single design variable.

These are fairly traditional assumptions for weight minimization problems. The effect of attachments and joints is ignored and attention is restricted to essentially membrane type of behavior. In a finite element idealization, only membrane plates and pin-ended frames are used to model the structure*.

The weight of the structure can then be written in the form

$$W = \sum_{i=1}^m \bar{W}_i A_i \quad (2)$$

For the constraint conditions, basically similar conditions must be satisfied:

- (3) A constraint is applied to a behavioral characteristic of each individual member of the structure or on the linear sum of contributions arising from each member.

*There is an exception to this wherein thin-walled circular tubes can be used to model space frames with flexural members.

- (4) Each constraint or constraint contribution must be expressible as an explicit differential function of the design variables.

Condition 3 clearly permits the consideration of member stresses and minimum sizes directly and is consistent with the classical virtual force method for calculating displacements.

Condition 4 is of much more complex significance and has a dominant influence on the range of applicability of the Lagrangian approach to structural optimization.

For minimum sizes, Condition 4 imposes no special restrictions. To be consistent with other constraints, it is found to be convenient to express minimum size constraints in the form

$$\frac{\bar{A}_i}{A_i} \leq 1 \quad i = 1 \text{ --- } m \quad (3)$$

where A_i is the current value of a member size

\bar{A}_i is its minimum allowable value.

and m is the number of elements in the structural model

The corresponding term in the Lagrangian is written

$$\sum_{i=1}^m \mu_i \left(\frac{\bar{A}_i}{A_i} - 1 \right)$$

where μ_i are the associated Lagrangian multipliers.

Displacement of a structure can be computed using the virtual force method.

For a pin jointed framework, this has the form

$$\delta_i = \sum_{i=1}^m \frac{S_i^P S_i^j L_i}{E_i A_i} \quad (4)$$

where S_i^P are the actual element forces due to the applied loading
 S_i^j are virtual element forces associated with a virtual unit force applied at the point of the unknown displacement δ_j

L_i and E_i are the length and moduli of the bar elements

For plate elements, similar expressions can be derived. The more general form of Equation 4 can be written

$$\delta = S^{Pt} f S^j \quad (5)$$

where S^P, S^j are the vectors of element forces

and f is the assembled diagonal element flexibility matrix

Although the displacement of Equation 4 has been derived here through what is basically a force method formulation, this is not an essential requirement. It can be obtained with the same degree of validity using a displacement method.

In Equation 4 S_i^j need only be in static equilibrium with the virtual load and is hence independent of the member sizes A_i . On the other hand, the basic force distribution S_i^P is independent of A_i 's only in statically determinate structures. For more generally redundant systems, S_i^P is an almost transcendental function of all member variables. As such, the displacement constraint could not meet the requirement of condition 4 for differentiation. In early optimality criteria studies it was assumed that S_i^P was independent of A_i to overcome this obstacle. The argument used was that the internal force distribution, even in a redundant structure, was only weakly influenced by the elastic characteristics of the members and had negligible variation for small changes in member size. Unfortunately, this is not generally true and, as will be discussed later, the error in this assumption has hitherto proved to be the major stumbling block in introduction of stress constraints. Fortunately, for the case of displacement constraints, the assumption is found to introduce no error. The reason is that the assumption of invariant S_i^P ignores terms in the differentiation of the type $\left(\frac{\partial S_i^P}{\partial A} \delta s^j \right)$. It has been shown⁽⁵⁾ that these terms do vanish over the entire structure, since they represent the virtual work of self-equilibrating systems and are hence zero.

Hence for displacement, the constraint condition can be written in the form

$$\sum_{i=1}^m \frac{e_i^j}{A_i} \leq \bar{c}_j \quad j = 1 \dots n \quad (6)$$

where $\bar{e}_{ij} = \frac{S_i^P S_i^j L_i}{E_i}$ are now considered to be

independent of the variables A_i

and \bar{C}_j is the allowable value of the j^{th} displacement constraint

The corresponding terms in the Lagrangian are

$$\sum_{j=1}^n \lambda_j \left(\frac{1}{\bar{C}_j} \sum_{i=1}^m \frac{\bar{e}_{ij}}{A_i} - 1 \right)$$

where λ_j are the associated Lagrangian multipliers.

For stress constraints, the form is deceptively simple.

$$\sigma_i = \frac{S_i}{A_i} \quad (7)$$

As discussed above, this will not meet the requirements of condition 4 directly for redundant systems, since S_i is a too complex function of all A_i . There is no corresponding term for stress constraints to that which allowed the derivative $\frac{dS}{dA}$ to be ignored in the displacement constraint derivation. If it is assumed that S_i is invariant, which is exact for statically determinate structures only, the use of the optimality criteria approach will lead directly to a fully stressed design -- which is, again, exact for statically determinate structures only.

The only solution to this problem is to include either directly or indirectly terms which will allow representation of $\left(\frac{\partial S}{\partial A} \right)$. One

method is to compute finite difference approximations but this is generally too expensive computationally. Another approach lies in the use of the force method as will be discussed in the next section.

Omitting stress constraints directly reduces the problem to minimum size and displacement constraints only.

The Lagrangian can now be written as

$$L = \sum_{i=1}^m \bar{W}_i A_i + \sum_{i=1}^m \mu_i \left(\frac{\bar{A}_i}{A_i} - 1 \right) + \sum_{i=1}^m V_i^2 \left(\frac{S_i}{C_i A_i} - 1 \right) + \sum_{j=1}^n \lambda_j \left(\frac{1}{C_j} - \sum_{i=1}^m \frac{C_{ij}}{A_i} - 1 \right) \quad (8)$$

The third term of Equation 8 for stress constraints can only be strictly used if the structure is statically determinate. The term is included here for completeness. Setting the differentials of Equation 8 with respect to A_i equal to zero, leads to m non-linear equations with the design variables A_i and the Lagrangian multipliers μ_i and λ_j as unknowns. Selection of a set of active constraint conditions will define (a) which Lagrangian multipliers are zero-corresponding to inactive (inequality) constraints and (b) provide a set of k (where $k \leq m$) non-linear equations with design variables as unknowns.

If the set of active constraints is known accurately and $k = m$, the constraint equations can be solved using a Newton-Raphson

approach to yield the optimum design. As will be discussed later this set of circumstances does not occur too frequently. Either $k \neq m$ or, even more likely, the active set of constraints is unknown.

If only one displacement constraint is active, the resulting problem, although still nominally non-linear, can be solved algebraically. This was the basis for the envelope method⁽¹⁾ from which the OPTIM programs have been developed. In the envelope approach, each displacement constraint was considered in isolation and an optimum design generated to satisfy that constraint alone. The interaction of several simultaneously active constraints was approximated by selecting a composite design from the envelope of the individual designs. This composite design was modified using a concept of active and passive member partitioning wherein the design of members was allocated to be controlled by a specific constraint condition. This partitioning was further modified by a representation of member minimum sizes and stresses. The stresses were effectively represented by fully stressed design criteria. While the envelope method has achieved some success, its greatest weakness still lies in the area of stress constraints.

Alternate approaches to the approximate solution of the non-linear equations arising from the differentiation of Equation 8 have been very thoroughly investigated. Reference 5 provides a comprehensive comparison of these methods of iterative solution of the non-linear problem and attempts to identify the parameters which affect the erratic convergence characteristics frequently encountered.

The goal of the present work was not to duplicate these efforts but to pursue the more elusive goal of incorporating stress constraints in a vigorous manner. The key to this was felt to lie in the philosophy of the force method of analysis.

2.2 Fundamentals of Force Method

Prior work had indicated the use of the force method philosophy did provide the basis for the vigorous representation of stress constraints in a Lagrangian formulation.

In the vast majority of early work in the field of structural optimization, the principal effort was directed towards the development of methods of redesign per se. The use of a finite element method of analysis was considered merely as an adjunct for determining stresses and displacements rapidly and with a minimum of computational effort. It was stated that any other method of analysis (e.g., finite difference) would be equally suitable. There was no attempt (or indeed any real reason) to integrate the analysis and redesign philosophies in any real way. With the dominance of the displacement method of finite element analysis in the 60's and 70's this method was fully developed and readily available for incorporation into any optimization program with a minimum of effort. The force method which had achieved initial prominence in the 50's was effectively discarded because of some inherent disadvantages it was felt to display in comparison with the apparently simpler displacement method.

In prior work on optimization^(6), some use was made of the force method, but this was primarily directed at the reduction of computational effort associated with iterative analysis. The analysis was not made an integral part of the optimization procedure. Advantage was taken of the form of the equations to be solved in the iterative analysis to reduce computational times considerably compared with displacement method techniques.

Since there is a general unfamiliarity with the force method it is appropriate to discuss and define the concepts which will be used in subsequent sections.

The force method of analysis is based upon the overall enforcement of structural equilibrium and the subsequent satisfaction of compatibility*.

For a general finite element model of a structure a set of overall equilibrium equations can be written

$$[P] = [A] [S] \quad (9)$$

which relate the externally applied loads $[P]$ and the internal member forces $[S]$. If the number of equilibrium conditions is equal to the number of internal forces the structure is statically determinate and $[A]$ is a square non-singular matrix. If the number of equilibrium conditions is less than the number of unknown forces, the structure is indeterminate

*In the simplest of forms the displacement method can be seen to be the converse of this concept.

or redundant; the degree of redundancy being defined by the excess of unknown forces over the number of equilibrium conditions. It is the latter case which is of significance.

The internal force distribution S is assumed to be the sum of two component distributions. The first component is one which is in static equilibrium only with the applied loading system while the second component arises from internally self-equilibrating force systems of undetermined magnitudes.

Expressed mathematically

$$S = b_0 + b_1 X \quad (10)$$

where b_0 is in static equilibrium with the applied loading
and b_1 are values of self-equilibrating unit force systems
whose magnitudes X will be determined

It is clear that both b_0 and b_1 systems will of necessity, individually violate compatibility conditions, but their weighted linear combination will ensure satisfaction of compatibility. The force distributions b_0 and b_1 are not generally unique, except in very simple structures. Their determination may require use of a concept referred to as a structure cutter.

From the definitions of the b_0 matrix it is clear that there must be zero forces in as many members as there are redundancies. These members can be considered as being cut in the basic b_0 system*.

*These cuts are not real but are expressed as such to give physical illustration to the concept of compatibility violation.

The b_0 matrix can be written as

$$b_0 = \begin{bmatrix} b'_0 & P \\ -0 & - \end{bmatrix} \quad (11)$$

where b'_0 are the values of the element forces due to unit values of P .

In the 'cut' members the only forces will be those due to the redundant systems. If a redundant system is assigned a unit magnitude in a 'cut' member, the b_1 matrix may be written as

$$b_1 = \begin{bmatrix} b'_1 \\ -I \end{bmatrix} \quad (12)$$

where the partitioning is as in Equation 11. Combining Equations 11 and 12 into 10 yields

$$[S] = \begin{bmatrix} b'_0 & b'_1 \\ -0 & -I \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix} \quad (13)$$

This matrix is square and invertible and its inverse may be written

$$\begin{bmatrix} P \\ X \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -0 & -I \end{bmatrix} [S] \quad (14)$$

Equation (14) corresponds to two equations, the second of which is trivial. The first is exactly Equation 9 which defines overall system equilibrium.

Thus to generate the b_0 and b_1 matrices it is only necessary to start with the equilibrium condition, Equation 9. From the rectangular matrix $[A]$ a non-singular square matrix $[A_1]$ is extracted

using a suitable rank technique. The b_0 and b_1 matrices are then given by

$$b_0 = \begin{bmatrix} A_1^{-1} \\ 0 \end{bmatrix} \quad \text{and} \quad b_1 = \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \quad (15)$$

The extraction of $[A_1]$ is not a unique process and can be influenced by a number of factors. The choice of the b_0 matrix is of minor significance, but the b_1 matrix will greatly affect the conditioning of equations to be solved in satisfaction of compatibility. A considerable effort⁽¹⁰⁾ has been devoted to the development of structure cutters which improve conditioning but their study is outside the scope of the present discussion.

To enforce compatibility, the magnitudes of the redundant forces must be selected to ensure that the relative displacements of the cuts in the structure be set to zero. (An analogy to structural optimization wherein displacements are constrained to a prescribed value can be observed.) To accomplish this the virtual force method is used; the virtual force distribution being given by the b_1 matrix.

In matrix notation the compatibility condition is given by

$$b_1^t f^s S = 0 \quad (16)$$

or
$$b_1^t f^{b_0} + (b_1^t f^{b_1}) X = 0$$

$$\text{Hence} \quad X = -(b_1^t f^{b_1})^{-1} (b_1^t f^{b_0}) \quad (17)$$

Equation 17 is the classic expression for the value of the redundancies. While the apparent complexity of Equation 17 may indicate computational cost, it must be pointed out that the order of the matrix $(b_1^t f b_1)$ is only equal to the number of redundancies -- a number usually considerably smaller than the corresponding number of degrees of freedom encountered in the solution of a displacement method approach.

Having discussed the force method concepts, it is now appropriate to examine its relation to optimization.

2.3 Role of Force Method in Structural Optimization

In Section 2.1, the optimality criteria approach to structural optimization was discussed. The applicability of the Lagrangian formulation for minimum member sizes, displacement constraints and stress constraints in determinate structures was presented.

In the force method analysis, the internal force distributions b_0 , b_1 were determined purely from static considerations. Hence, apart from the final satisfaction of compatibility conditions, the entire analysis effectively treats only a statically determinate system. Reference to Equation 14 indicates very clearly that if the redundant forces X are regarded as part of the applied force set, the entire system is indeed statically determinate. Thus instead of considering a redundant structure with an external load system P applied, the structure can equally be viewed as a determinate system with an external loading system $\begin{bmatrix} P \\ X \end{bmatrix}$. The only additional requirement is that this structure will be designed to

ensure that the displacements associated with the (unknown) forces X shall be constrained to zero values. In an optimization sense stress constraints may now be considered because of the static determinancy.

Clearly this transformation from redundancy to determinancy cannot be accomplished merely by making the above statements. Some additional terms must be introduced into the mathematical formulations. What has happened is that the set of design variables must be expanded to include the values of the redundancies as unknowns.

Expressions for the internal forces must now be written as linear functions of the additional variables X (Equation 10) .

Condition 4 is still satisfied since the new functional relationships are still linear. The set of optimality criteria must be expanded by the generation of not only $\frac{\partial L}{\partial A}$ but also $\frac{\partial L}{\partial X}$ conditions.

The full set of optimality criteria and constraint conditions applicable to a force method formulation is presented in Fig. 1. In the derivation of the equations of Fig. 1, two types of displacement constraints have been considered, those associated with the compatibility conditions and those with externally defined displacement limitations (Fig. 9). The major difference between the two terms lies principally in the absence of the unity for the compatibility constraint value. Thus the stress constraint problem has been solved effectively by transformation into a type of displacement constraint. It is recognized that an alternate approach would have been to express each individual (bar) element stress constraint as a relative displacement

LAGRANGIAN

$$L = \sum_{i=1}^m \bar{w}_i A_i + \sum_{i=1}^m \left(\frac{A_i^*}{A_i} - 1 \right) + \sum_{i=1}^m v_i \left[\left(\frac{B_o^i + \sum B_1^{ij} x_j}{A_i \sigma_i} \right) - 1 \right] \\ + \sum_{j=1}^p \lambda_j \left[\sum_{i=1}^m \left(B_o^i + \sum_{k=1}^p B_1^{ik} x_k \right) \bar{f}_i B_1^{ij} \frac{1}{A_i} \right]$$

OPTIMALITY CRITERIA

$$\frac{\partial L}{\partial A_i} = 0 = \bar{w}_i - v_i \frac{A_i^*}{A_i^2} - v_i \left(\frac{B_o^i + \sum B_1^{ij} x_j}{A_i^2} \right) - \sum \lambda_j \frac{1}{A_i^2} \left(B_o^i + \sum B_1^{ik} x_k \right) \bar{f}_i B_1^{ij} = f_{A_i}$$

$$\frac{\partial L}{\partial x_j} = 0 = \sum v_i \frac{B_1^{ij}}{A_i \sigma_i} + \sum \lambda_k \sum \frac{B_1^{ij} B_1^{ik} \bar{f}_i}{A_i} = x_j$$

CONSTRAINT CONDITIONS

Note: \bar{f}_i = Flexibility matrix

$$\text{MINIMUM AREA} \quad \frac{\partial L}{\partial \mu_i} = 0 = \left(\frac{A_i^*}{A_i} - 1 \right) = f_{A_i}$$

$$\text{MAXIMUM STRESS} \quad \frac{\partial L}{\partial v_i} = 0 = \left(\frac{B_o^i + \sum B_1^{ij} x_j}{A_i \sigma_i} \right) - 1 = f_{\sigma_i}$$

$$\text{COMPATIBILITY} \quad \frac{\partial L}{\partial \lambda_j} = 0 = \sum \left[\left(B_o^i + \sum B_1^{ik} x_k \right) \bar{f}_i B_1^{ij} \frac{1}{A_i} \right] = f_{\epsilon_j}$$

FIGURE 1 - OPTIMALITY CRITERIA AND CONSTRAINT CONDITIONS

constraint at the two ends. This idea has been used⁽⁹⁾, but it is felt to have the disadvantage of requiring more computational effort for reasons which will be discussed in Section 3.8, and also its extension to other than bar-type elements is unclear. The definition of a simple relative displacement constraint criterion will not usually suffice for a multi-node plate element.

Having now the rigorous formulation which includes stress, displacement, and minimum size constraints, a method of solving the non-linear equations must be derived.

2.4 Equation Solution and Constraint Detection

In general, there are potentially many more constraints than variables and the actual number of active constraints is a small subset of the total number present. An inactive constraint is represented by a zero value of the corresponding Lagrangian multiplier and the term hence vanishes from Equation 8. An active constraint corresponds to a non-zero multiplier. Although no physical meanings are usually attached to the values of the Lagrangian multipliers, it is clear that their values are of great significance.

Consider the first two equations of Figure 1, e.g., L and $\frac{\partial L}{\partial A_i}$. Examination of the form of L and $\frac{\partial L}{\partial A_i}$ reveals they can be written in the abbreviated forms

$$L = \sum_i \bar{W}_i A_i + \sum_k \lambda_k \left(\sum_i \frac{K_i}{A_i} - 1 \right) \quad (18)$$

+ \sum compatibility terms

$$\frac{\partial L}{\partial A_i} = \bar{W}_i - \sum_k \lambda_k \frac{K_i}{A_i} + \sum \text{compatibility terms} \quad (19)$$

where λ_k represents all multipliers

and K_i represents the remaining terms.

By linear combination, the following relationship can be formed

$$L + \sum_i A_i \frac{\partial L}{\partial A_i} = 2 \sum_i \bar{W}_i A_i - \sum_k \lambda_k \quad (20)$$

For the optimum structure all terms in L, except the first, vanish. Hence

at the optimum $L^* = W^*$, where an asterisk indicates an optimal value.

Also for the optimum structure $\frac{\partial L}{\partial A_i} = 0$. Therefore Equation 20

reduces to

$$\begin{aligned} W^* &= 2W^* - \sum_k \lambda_k^* \\ \text{or} \quad W^* &= \sum_k \lambda_k^* \end{aligned} \quad (21)$$

Thus at the optimum the weight of the structure is given by the sum of the active constraint Lagrangian multipliers.*

While this summation does indicate a physical meaning, the only major use which has been made of Equation 21 has been as an

*It is to be noted that Lagrangian multipliers associated with compatibility constraints do not contribute to summation of multipliers which equals the weight. This is due to the absence of the unity which would survive in the combination of Equations 19 and 20.

aid in determining when convergence on an optimal design has occurred.

Since the form of the non-linear equations to be solved is well defined, it was felt that a Newton-Raphson method to solution was a most practical approach. In a N-R formulation, derivatives of all the functions are required. Since the entire approach via the force method has resulted in explicit terms of relatively simple algebraic form, the creation of the derivatives required for the N-R solution is a relatively straightforward matter. The calculation of these derivative terms is exact, which may assist in convergence, but more importantly, their evaluation does not require finite difference or similar calculations which would involve repeated structural analyses.

For the present set of nonlinear equations consisting of optimality criteria and constraints, the linearized N-R equations are given in Figure 2. All the derivatives are explicitly expressible in terms of the constituent matrices. The derivatives are listed in Figure 3.

In simple test problems, where the active constraint population was known from the start, the N-R procedure has been shown to converge rapidly on the optimal solution. In a small problem with three structural elements and two displacement type of constraints, the envelope method required 54 iterations to achieve a converged solution. The N-R approach converged in 5 iterations on a design slightly lighter (by 0.3%) than that obtained by the envelope method.

Optimality Criteria

$$\frac{\partial f_{A_i}}{\partial A_i} \delta A_i + \sum_{j=1}^P \frac{\partial f_{A_i}}{\partial X_j} \delta X_j + \frac{\partial f_{A_i}}{\partial \mu_i} \delta \mu_i + \frac{\partial f_{A_i}}{\partial v_i} \delta v_i + \sum_{j=1}^P \frac{\partial f_{A_i}}{\partial \lambda_j} \delta \lambda_j = -f_{A_i}$$

$$\sum_{j=1}^m \frac{\partial f_{X_j}}{\partial A_i} \delta A_i + \sum_{j=1}^m \frac{\partial f_{X_j}}{\partial v_i} \delta v_i + \sum_{k=1}^P \frac{\partial f_{X_j}}{\partial \lambda_k} \delta \lambda_k = -f_{X_j}$$

Constraint Conditions

$$\frac{\partial f_{A_i}}{\partial A_i} \delta A_i = -f_{A_i}$$

$$\frac{\partial f_{S_i}}{\partial A_i} \delta A_i + \sum_{j=1}^P \frac{\partial f_{S_i}}{\partial X_j} \delta X_j = -f_{S_i}$$

$$\sum_{j=1}^m \frac{\partial f_{E_j}}{\partial A_i} \delta A_i + \sum_{k=1}^P \frac{\partial f_{E_j}}{\partial X_k} \delta X_k = -f_{E_j}$$

FIGURE 2 - LINEARIZED NEWTON-RAPHSON EQUATIONS

$$\frac{\partial f_{A_i}}{\partial A_i} = \frac{2\mu_i A_i^*}{A_i^3} + 2\nu_i \left(B_0^i + \frac{\sum_{j=1}^p B_i^{ij} X_j}{A_i^2 \sigma_i} \right) + \sum_{j=1}^p 2\lambda_j \frac{1}{A_i^3} \left[(B_0^i + \sum_{k=1}^p B_i^{ik} X_k) \bar{f}_i B_i^{ij} \right]$$

$$= \frac{2(\bar{w}_i - f_{A_i})}{A_i}$$

$$\frac{\partial f_{X_j}}{\partial A_i} = -\frac{\nu_i B_i^{ij}}{A_i^2 \sigma_i} - \sum_{k=1}^p \lambda_k \frac{B_i^{ik} B_i^{ij} \bar{f}_i}{A_i^2} = \frac{\partial f_{A_i}}{\partial X_j}$$

$$\frac{\partial f_{A_i}}{\partial \mu_i} = -\frac{A_i^*}{A_i^2} = -\frac{\partial f_{A_i}}{\partial A_i} \quad \frac{\partial f_{A_i}}{\partial \nu_i} = -\frac{B_0^i + \sum_{j=1}^p B_i^{ij} X_j}{A_i^2 \sigma_i} = \frac{\partial f_{S_i}}{\partial A_i}$$

$$\frac{\partial f_{A_i}}{\partial \lambda_j} = -\frac{(B_0^i + \sum_{k=1}^p B_i^{ik} X_k) \bar{f}_i B_i^{ij}}{A_i^2} = \frac{\partial f_{\epsilon_j}}{\partial A_i} \quad \frac{\partial f_{X_j}}{\partial \nu_i} = \frac{B_i^{ij}}{A_i \sigma_i} = \frac{\partial f_{S_i}}{\partial X_j}$$

$$\frac{\partial f_{X_j}}{\partial \lambda_k} = \sum_{i=1}^m B_i^{ik} B_i^{ij} \bar{f}_i \frac{1}{A_i} = \frac{\partial f_{\epsilon_j}}{\partial X_k}$$

FIGURE 3 - NEWTON-RAPHSON DERIVATIVES

With the selection of the N-R as the prime solution technique, the major remaining task appeared to be the generation of a method for differentiating between active and inactive constraints.

Since all constraints are either satisfied as equalities or their corresponding Lagrangian multipliers vanish, the inequalities associated with all constraints can be transformed into equalities by expressions including their multiplier. Thus a minimum size constraint would be written

$$\mu_i \left(\frac{\bar{A}_i}{A_i} - 1 \right) = 0 \quad (22)$$

The resulting set of equations tends to become very large, but was indeed a set consisting of equalities only. For a problem with m elements, p redundancies and constraints on both stresses and minimum sizes, a total of $3m + 2p$ equations are required.

The first problem to be considered was the selection of suitable starting points for the linearized search. During the course of this stage of program development, a considerable number of strategies were tested, with very mixed results. The conclusion drawn at that time was that the search procedure was extremely sensitive to the initially selected values - particularly the values chosen for the Lagrangian multipliers. There was some evidence, in a selected number of cases, that areas tended to converge on known optimal solutions fairly rapidly, while multipliers took many more iterations to stabilize. From later evidence, accumulated in future

developments, it became clear that these were somewhat erroneous conclusions. Since the equations are linear in the multipliers, the actual starting values selected for the multipliers is immaterial. What is of dominant importance is their population--i.e., which are zero and which are non-zero. Of lesser importance are the starting values for the member sizes.

Since these points were not fully recognized, various schemes for selecting starting values were investigated. For example, since the sum of the multipliers was known to equal the weight, at the optimum, the following scheme was devised. It is assumed that the applied load is carried entirely in the basic b_0 structure, and its members are sized accordingly for stress limits, the other members are set to minimum areas. Now for each member, the individual weight is computed and half this value is assigned to the respective minimum area and stress Lagrangian multipliers. This approach assumes, a priori, that every potential constraint is active and has its own non-zero multiplier. As an alternate form of this starting point, again the member weights are used to compute multipliers, but the full member weight is assigned to either constraint dependent upon which is assumed dominant.

These approaches and others did achieve some modicum of success on three-bar truss problems - particularly those discussed in proposed contractor tasks.

The Newton-Raphson procedure was found to be relatively unstable, and extensive move-limits had to be introduced to prevent

divergence occurring in early iterations. Eventually, a limit of $\pm 30\%$ variation in a design variable was found to be a satisfactory limit for smaller problems. The logic of the program was designed to capture additional constraints, should violations occur and to release them, if the associated multiplier should become negative, or if the potential move would take the design away from a constraint.

The 10-bar truss problem with the high allowable stress in member 10, was next investigated. This proved to be quite troublesome and required a considerable amount of numerical experimentation to obtain solutions. Putting in a design close to the optimal did permit convergence fairly rapidly, thereby demonstrating overall correctness of the coding.

For cases in which $\sigma_{10} < 37500$ psi, the optimal design is fully stressed and the procedure did finally achieve the correct solutions in a limited number of iterations - less than would have been required using a stress-ratio method. For $\sigma_{10} > 37500$ psi the convergence was very poor and sometimes totally unstable.

The known optimal solution was reviewed and two potential problem areas were tentatively identified. Firstly, there were members present which were both fully stressed and at minimum area and secondly, the optimum design had, by happenstance 11 active constraints. The former condition was no real novelty since it had been encountered in some of the 3-bar test problems used previously. The latter condition, which arose from the geometry and loading of the structure was felt to be potentially more serious.

Further examination of the cases which failed to converge did indicate that the set of linearized equations being solved were probably singular - or very close to it. Round-off accumulation prevents exact singularity occurring but examination of the values of determinants suggest the presence of this problem.

The possibility of over-constraint was felt to be a reality and hence attempts were made to provide for detection and elimination of singularities arising from this source.

Recourse was made to the use of the Cholesky method of equation solution since the array of equations is symmetric. In the triangular decomposition stage, a singularity appears as a zero (or near zero) term on main diagonal. When such a term is identified, the associated equation can be eliminated from the array by a decoupling process. This has the effect of winnowing out equations which are linearly dependent on any combination of previous equations. Thus the order of the equations will affect which equations are discarded.

Some analysis of the algebraic form of the equations for three-bar trusses did indicate circumstances under which singularities could occur - although no generalization was possible. It was finally concluded that it was preferable to discard constraints associated with minimum sizes rather than stresses. This latter decision was purely arbitrary and was based upon a judgment that stress constraints were of greater importance than minimum sizes.

The Cholesky equation solver was modified accordingly to remove singularities and the ordering of the equations was changed to ensure that stress constraints were favored. These changes had no impact on the small three-bar test problems but did affect the 10-bar problems.

Some degree of success was attained but the convergence was still found to be highly dependent on the starting point selected. Some further effort was expended on modifications to step size limits but without much change in the situation.

Finally, the form of the equations was reviewed again and it was determined that the strategies used had created an undesirable situation. If any member reached its lower limit, it would never be possible on any subsequent iteration for that member to increase in size and move away from the constraint. That is, once a minimum size constraint became critical, it was retained. This was clearly unacceptable and required a review of the entire approach.

From consideration of the array of governing equations, it was recognized that the non-linearity was confined to the primary variables only, the secondary (Lagrangian) variables were purely linear. Although this was considered at some length, no use could be made of this fact until Reference 11 became available. This work had the effect of redirecting the entire effort into a more fruitful avenue of approach.

2.5 Linear Programming

While the N-R solution technique combined with the force method does provide the potential for the vigorous incorporation of stress constraints, it is clear that the major problem of constraint detection still remains.

From examination of the form of the equations being considered, it can be seen that the nonlinearities arise from the design variables. The equations are purely linear in the Lagrangian multipliers. While this suggests that some form of decomposition into linear and non-linear problems might be possible, no progress was made in simplifying solution techniques until the work presented in Reference 11 not considered.

In Reference 11 the concept of duality is introduced and is used to linearize the problem. While the duality approach is entirely valid, the same linearization can be achieved in a more direct manner.

In Equation 21 it was demonstrated that the optimum weight is the sum of the Lagrangian multipliers associated with all non-zero value constraints. That means the optimum weight is explicitly independent of the (primal) design variables. Hence if design variables are selected which satisfy the constraint conditions either as equalities or inequalities, in an arbitrary manner, the optimality criteria are now a non-square set of $(m + p)$ linear equations with n unknowns ($n > m + p$).

Solution of this problem can now be accomplished using a standard linear programming technique (LPT), with the merit condition (weight) being simply the sum of the individual multipliers. In the LPT, all Lagrangian multipliers, except those associated with the compatibility conditions, will be restrained to have values greater than or equal to zero.

The LPT will minimize the merit function by selecting which variables shall have non-zero values (active constraints) and which remain zero (inactive constraints). The result is a constraint population, with as many constraints as there are variables.

With this known constraint population, a new design may be generated readily, as will be discussed in a later paragraph.

This new design may not, in general, satisfy all the actual constraints in the problem and hence may have to be adjusted, e. g., areas defined by stress constraints may be smaller than minimum allowable sizes. Dependent upon some simple test criteria, this new design may again be used as the starting point for another iteration of the LPT. This cycle may be repeated until convergence occurs, in which case the optimum design has been generated.

It is of interest to note what will occur in purely stress constrained problems. If other than the force method concept is used, all displacement and compatibility related terms will vanish from the optimality criteria equations of Figure 1. Examinations of the remaining terms in the equations shows that the array uncouples into m individual problems, each involving

μ_i and γ_i only. The solution to this then indicates that either a μ_i (minimum area) or γ_i (maximum stress) is present in each member, i.e., the classic FSD solution. The presence of the compatibility related terms associated with the force method formulation ensures avoidance of this pitfall.

The LPT approach generates a potential constraint population at each iteration. It also generates values for the non-zero Lagrangian multipliers. Experience indicates that these values are of minor worth, but the population so defined is extremely relevant. The force method approach is ideally suited to the determination of a structure which satisfies a prescribed set of constraints. The proviso must be made that the number of constraints equals the number of variables, i.e., that the design being sought does lie at a full vertex in the design space. This is consistent with the design population prescribed by the LPT solution which will have selected a full vertex as optimal in the linearized design space. The consequences of the optimal structure not occurring at a full vertex will be discussed in subsequent sections.

In determining a structure which corresponds to a given (full) set of active constraints, the key lies in the derivation of the internal force distribution. An illustration of this is the usual method of generating an FSD via stress-ratio.

An initial guess is chosen, the structure analyzed, and areas adjusted to bring stresses to their critical values and/or to their minimum sizes. The process is simple, but many analysis iterations may be required

for convergence. This indeed is the method conventionally used in conjunction with a displacement analysis technique.

Assume now that the distribution of critical constraints is known a priori, i.e., which members are fully stressed and which have minimum areas. Using the force method approach, the compatibility equations are

$$(b_1^t f b_0) + (b_1^t f b_1)X = 0 \quad (23)$$

Using a slightly different notation, Equation 23 can be written in the form

$$\sum_{i=1}^m b_1^{ij} \frac{\bar{f}_i}{A_i} (b_0^i + \sum_k b_1^{ik} X_k) = 0, j = 1 \dots n \quad (24)$$

For members in which the stress is critical $\frac{b_0^i + \sum_k b_1^{ik} X_k}{A_i}$ is replaced

by σ_i . For the other members A is set to A^{\min} . All A-values are now eliminated and the only unknowns are the redundant forces X_k . Solution of the linear equations leads to the required internal force distribution which satisfies the prescribed FSD conditions--without iteration. The areas are obtained from the known stresses and the computed internal forces.

If the constraint distribution given by the LPT routine is other than FSD-type, the equations to be solved become nonlinear. A member may be both fully stressed and have a minimum area. Consequently another member size will be completely unrestrained. The presence of active displacement constraints will similarly leave some members undefined.

In such cases, the set of compatibility equations must be expended by force-stress-minimum area relationships and/or displacement relationships. In either case the number of equations and variables increases to (redundancies + undefined areas) and the equations become non-linear. For their solution a Newton-Raphson technique has always proved to be entirely adequate for problems in which the defined constraint population corresponds to a feasible design. It can occur that the LPT will propose an unfeasible constraint population. The method of handling this situation is discussed in the program development section.

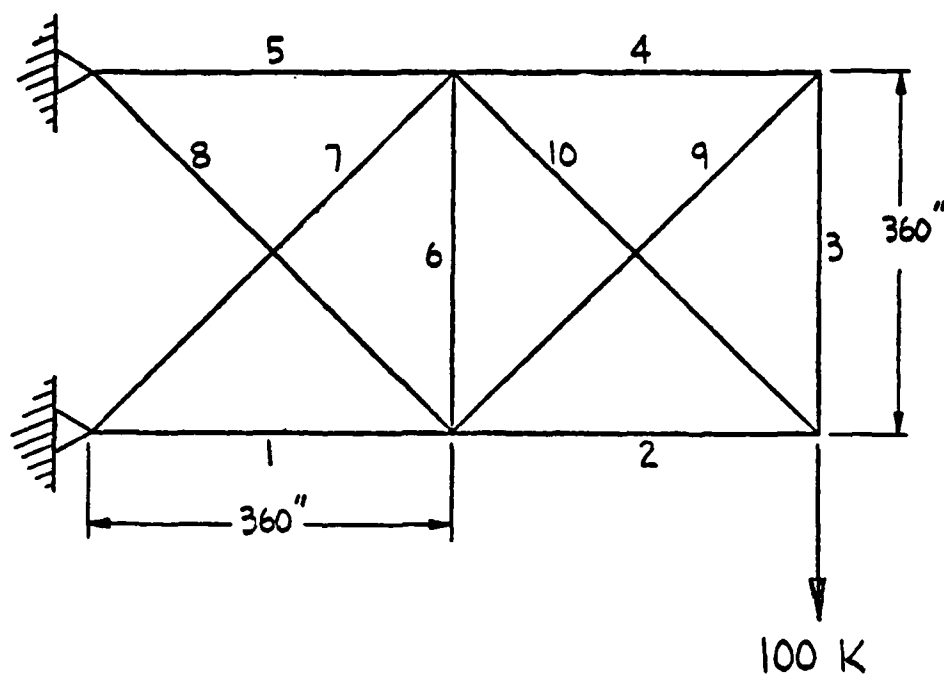
The LPT approach appeared to have great merit for overcoming the constraint definition problems described in the previous section. A new computer program was generated based upon the LPT and checked out on a variety of small 3- and 4-bar truss problems. The first major test of the program's effectiveness was the classic 10-bar truss problem, in which one member has a significantly higher allowable stress than the remaining nine members. This problem was extensively discussed in Reference 12, for a range of allowable stresses and had been subsequently used as an evaluation tool for different approaches to strength optimization. The new LPT approach was used on this problem and was found to be startlingly successful.

2.5.1 Ten-bar Truss Problem

The ten-bar truss shown in Figure 4 is constructed of aluminum alloy throughout ($E = 10^7$ psi, $\rho = 0.1$ lb/in³). The allowable stress in all bars except number 10 is ± 25000 psi. The allowable for member 10 has been varied in past problems from ± 25000 up to ± 75000 psi. It has been shown⁽¹²⁾ that for $\sigma'_{10} \leq 37500$ psi a FSD is optimum. For $\sigma_{10} > 37500$ psi the optimum design (weighing 1497.6 lb) is independent of σ_{10} and member 10 is neither critically stressed or at minimum size. Use of a stress ratio approach leads to a totally different design weighing 1725.24 lb, an error of approximately 15% with a completely different distribution of element areas.

The output from the pilot computer program applied to the 10-bar problem with $\sigma_{10} = \pm 75000$ psi, is shown in Table 1. Computer program development will be discussed in a later section, but it is appropriate to mention here some of the salient features of these output results.

No starting point design was specified so the program automatically selects minimum areas as the default option. Experience has indicated that such designs may be very poor approximations of optimal systems, hence one stress-ratio cycle is optionally permitted to generate a more-or-less feasible design. This design is still very different from either the (known) optimum or the converged FSD. This cycled design is used in the LPT routine and results in the proposed population of minimum areas and fully stressed members shown in Table 1. The negative sign on stressed



	A_{MIN} (IN ²)	σ_{MAX} (PSI)
MEM 1-9	0.1	25,000
MEM 10	0.1	75,000

FIGURE 4 - 10-BAR TRUSS, BASIC STRUCTURE

```

/INPUT
/JOB TIME=1.60
/INCLUDE RGH8C
/INCLUDE NEWT
/INCLUDE SUBS
/INCLUDE BARTEN
/ENDRUN

```

```

PAUSE ENTER TOL,TOL1,IRD,ISR,IFRINT,KODE1,KODE2
1.E-8,1.E-7,0.0,-3.0,0

```

STRESS RATIO AREAS

```

1 0.81854E 01 2 0.23950E 01 3 0.16050E 01 4 0.16050E 01 5 0.78146E 01
6 0.14196E 01 7 0.53946E 01 8 0.59190E 01 9 0.22698E 01 10 0.11290E 01

```

```

x
1 -0.21033E 05 2 0.30183E 05

```

```

MEM 1 2 3 4 5 6 7 8 9 10

```

```

AREA 0 0 0 1 0 1 0 0 0 0
STR -1 -1 1 1 1 0 -1 1 -1 0

```

```

X VALUES 1 -0.18533E 05 2 0.47683E 05 3 0.12012E 01
X VALUES 1 -0.12695E-01 2 0.97656E-03 3 0.17090E 00

```

N-R AREAS

```

1 0.81000E 01 2 0.39000E 01 3 0.10000E 00 4 0.10000E 00 5 0.79000E 01
6 0.10000E 00 7 0.55154E 01 8 0.57983E 01 9 0.14142E 00 10 0.36769E 01
X VALUES 1 -0.25000E 04 2 0.25000E 04

```

LAGRANGIAN MULTIPLIERS

```

1 0.0 2 0.0 3 0.0 4 0.60001E 01 5 0.0
6 0.36000E 01 7 0.0 8 0.0 9 0.0 10 0.0
11 0.33400E 03 12 0.78000E 02 13 0.66000E 02 14 0.60000E 02 15 0.22200E 03
16 0.0 17 0.15600E 03 18 0.42000E 03 19 0.13200E 03 20 0.0
21 0.69333E 02 22 -0.69333E 02

```

```

ITER NO. 1

```

```

SUM MEM WGT = 0.14976E 04 SUM LAM = 0.14976E 04

```

```

M.0070 END
M.0072 BEGIN

```

TABLE 1 - 10-BAR TRUSS - COMPUTER RESULTS

members indicates a compressive allowable. It is to be noted that member 10 is undefined, while member 3 is both fully stressed and of minimum area.

A Newton-Raphson routine is used to generate a design corresponding to the specified constraints using the previous design as a starting point. Since the structure is doubly redundant and one member area is undefined, there are only three nonlinear equations and three variables (X_1 , X_2 and A_{10}). Only three iterations are required for convergence. With these new values of the redundant forces, the known stresses, minimum-sized members and an area for member 10, the new design is generated and printed out. It can be seen that member 4 is at minimum size but this is not recognized as an active constraint. This is a rare, but not unique, situation which arises from the geometry and loading at Gridpoint 3. Members 3 and 4 are effectively linked together. The presence of a surplus, unused constraint has no influence on the problem or solution technique. A shortage of constraints, on the other hand, is of major consequence as will be discussed later. For this design, the corresponding non-zero Lagrangian multipliers can be obtained from the linear solution of the optimality criteria equations. Of the printed set, the first ten multipliers are those associated with minimum sizes, the next ten with stresses and the final two are the compatibility constraints.

Since all multipliers associated with non-zero type constraints have positive values, the design is optimal. As an additional check the sum of the multipliers does equal the weight. This solution, with its extremely

rapid convergence is an excellent demonstration of the power of the L. P. / Force method. The number of steps was minimal as was the computational effort.

The next stage in the check-out process was the application of the program to a larger problem -- the 22-bar double truss.

2.6 Twenty-two Bar Truss Problem

The 22-bar truss problem is an extension of the 10-bar problem insofar as it consists of two 10-bar cantilever frames connected at their extremities by a pair of hangars (Figure 5). A single load of 100,000 lbs is applied vertically at the extremity of the hanger. While the modulus is held constant at 10^7 psi throughout, various allowable stresses and material densities were to be considered. The truss problem has proved to be a significant one and has been investigated by a number of researchers^(9, 13) Due to a desire to explore the effects of varying critical parameters and also as a result of confusion as to specified values of some parameters, a number of cases have come into being. This confusion has been compounded, possibly by the fact that all problems have been labeled either Case A or Case B. Whatever particular values for stress allowables, densities or minimum sizes, the essence of the problem here is to demonstrate the manner in which the applied load is distributed optimally as the strength/density ratios are varied between the two half (10-bar) frames. The stress-ratio method is clearly useless for this type of problem since it includes no reference whatever to material density--clearly a major factor.

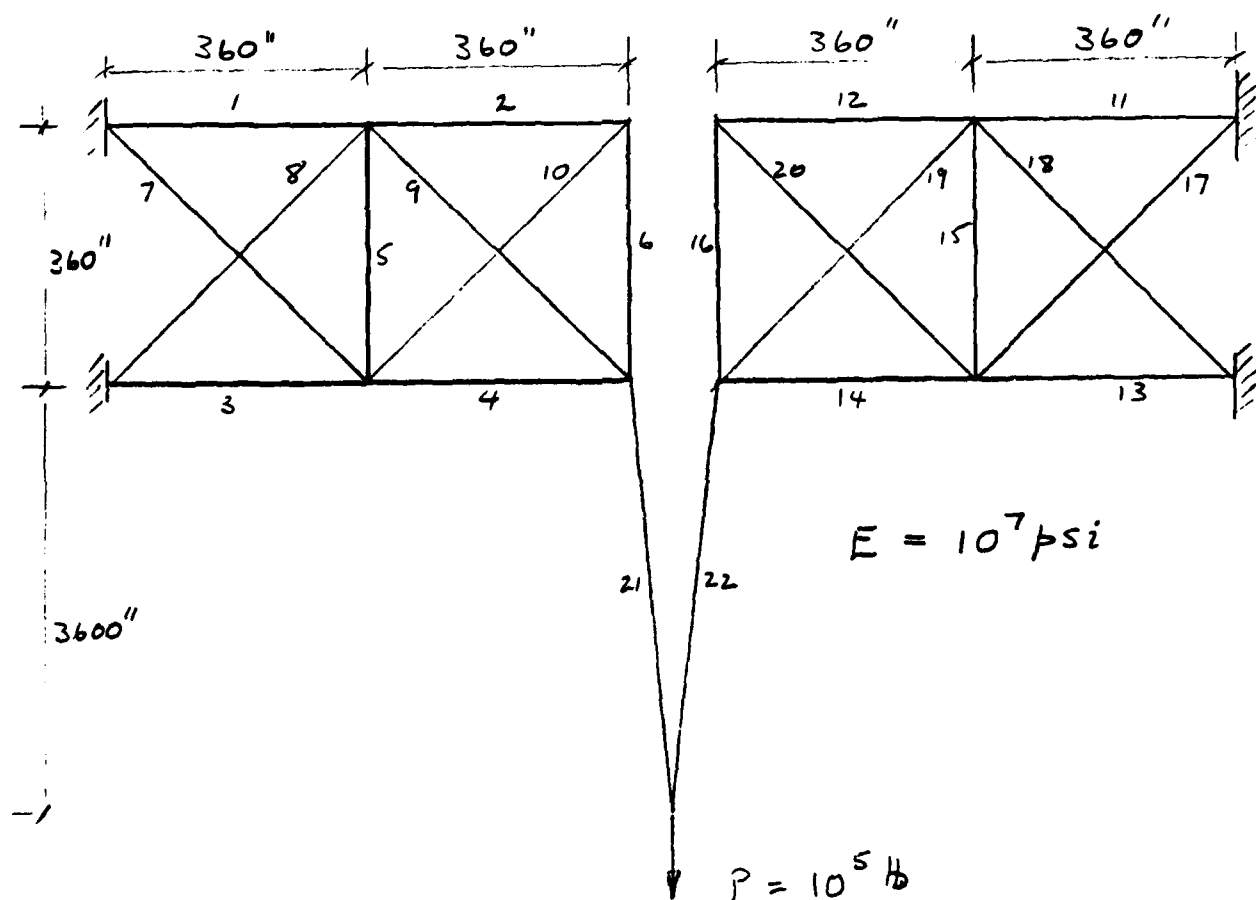


FIGURE 5 22-BAR TRUSS

Some attempts were made to run some cases using the program developed previously. The results obtained were extremely variable. Some cases produced solutions readily, whereas others failed to converge at all. This led to a re-evaluation of the concepts used. In particular, the question of the number of active constraints at the optimum design was considered. In all stress constraint test problems which could be reviewed either the number of active constraints was equal to the number of variables (full vertex) or it was impossible to state reliably how many constraints were active. This latter situation occurs in solutions where, for example, many stresses or areas are within a few percent of their critical values and there is no overt determination of the active constraint population.

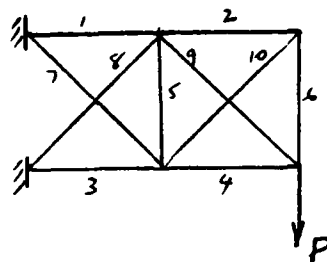
Since this question of number of active constraints is of prime consideration, a more detailed study of the 22-bar problem was deemed appropriate. Although this structure (in its various forms) has been difficult to optimize using general purpose optimization methods, the generation of optimal solutions analytically is relatively straight forward.

In all the cases of interest, the allowable stresses and densities are uniform throughout each 10-bar truss which form the two principal halves of the structure. The approach to the analytical optimization is through substructuring. The individual 10-bar trusses are optimized parametrically in terms of the applied load and the two halves are coupled through a compatibility condition. This approach is only possible here due

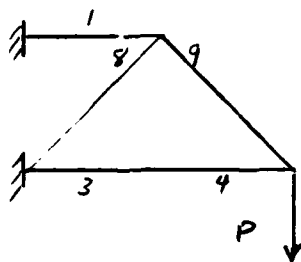
to the special geometry of the structure, although the substructuring concept has been proposed for general use in the optimization of complex structures⁽¹⁵⁾.

From the work performed in the study of the 10-bar truss, discussed previously, it has been conclusively demonstrated that a fully constrained design is optimal for uniform material and strength properties. For a vertical load applied at the outermost lower vertex, the optimal population has been determined. The population is that which normal engineering intuition would suggest, inasmuch as there is little load diffusion from principal determinate load paths when the redundant elements are introduced.

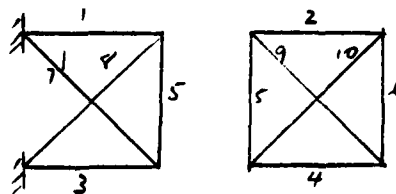
To optimize the structure, the force method approach is used.



a) Basic Structure



b) Determinate (b_0) System



c) Redundant (b_1) Systems

FIGURE 6 - 10-BAR TRUSS

The basic structure is shown in Figure 6 a). The constituent determinate b_0 and redundant b_1 systems are shown in b) and c). The choice of these systems is non-unique, but the members retained in the b_0 -system are those which form the natural primary load bearing structure. Provided the ratio between the applied load and the stress allowable is sufficiently high, the members 1, 3, 4, 8 and 9 present in the b_0 -system are fully stressed. The remaining members 2, 5, 6, 7 and 10 have minimum areas. It is possible that the applied load could be so small that some or all of the primary b_0 -system members would have to be set to their minimum areas instead of being fully stressed. In the present analysis, this situation has not been found to occur. For the systems shown, the relevant matrices are

$$b_0 = P \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \quad b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ -\sqrt{2} & 0 \\ -\sqrt{2} & 0 \\ 0 & -\sqrt{2} \\ 0 & -\sqrt{2} \end{bmatrix} \quad \bar{f} = \frac{L}{E} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad (25)$$

where \bar{f} is the diagonal matrix of unit sized element flexibilities

L is the length of a vertical or horizontal element ($\approx 360''$)

and E is the uniform elastic modulus.

With the information that members 1, 3, 4, 8, and 9 are fully stressed to the allowable value σ and the remaining members have a minimum area A^* , the usual compatibility conditions can be written as

$$\left. \begin{aligned} (2\sqrt{2} + 1)X_1 + X_2 &= -2\sigma A^* \\ X_1 + (3+2\sqrt{2})X_2 &= 3\sigma A^* \end{aligned} \right\} \quad (26)$$

$$\left. \begin{aligned} \text{Hence } X_1 &= -.68767264 \sigma A^* = \alpha_1 \sigma A^* \\ &\& X_2 = .63270460 \sigma A^* = \alpha_2 \sigma A^* \end{aligned} \right\} \quad (27)$$

The resultant internal force distribution, S, is

$$S = \begin{bmatrix} 2P + \alpha_1 \sigma A^* \\ \alpha_2 \sigma A^* \\ -P + \alpha_1 \sigma A^* \\ -P + \alpha_2 \sigma A^* \\ (\alpha_1 + \alpha_2) \sigma A^* \\ \alpha_2 \sigma A^* \\ -\sqrt{2} \alpha_1 \sigma A^* \\ -\sqrt{2} P - \sqrt{2} \alpha_1 \sigma A^* \\ \sqrt{2} P - \sqrt{2} \alpha_2 \sigma A^* \\ -\sqrt{2} \alpha_2 \sigma A^* \end{bmatrix} \quad (28)$$

The corresponding areas are

$$A = \begin{bmatrix} 2 P/\sigma + \alpha_1 A^* \\ A^* \\ P/\sigma - \alpha_1 A^* \\ P/\sigma - \alpha_2 A^* \\ A^* \\ A^* \\ A^* \\ \sqrt{2}(P/\sigma + \alpha_1 A^*) \\ 2(P/\sigma - \alpha_2 A^*) \\ A^* \end{bmatrix} \quad (29)$$

The weight of the optimized 10-bar truss is

$$W = [8P/\sigma + (3+2\sqrt{2} + 2\alpha_1 - 3\alpha_2)A^*] 360 \rho \quad (30)$$

where ρ is material density.

Thus the weight of each half-structure is expressible purely as a function of the load P at the free end. Since the total load on the structure is known, the problem is now reduced to a single variable optimization which defines the sharing of the externally applied load between the left and right-hand halves.

The problem is further simplified when the deflection of each half is computed. Using the unit virtual load method, only the members in the determinate b_0 system need be considered. Since these members are by definition fully stressed, their strains are independent of their areas, hence the total deflection is independent of the applied load.

$$\text{i.e. } \delta = \sum_{1,3,4,8,9} \sigma_f b_1 \quad (31)$$

Note here that σ is intended to signify both positive and negative stresses which must be accounted for in performing the summation.

Using the values of Equation 25 the deflection of the 10-bar frame is

$$\delta = \frac{8L\sigma}{E} \quad (32)$$

Since L and E are not variables in this problem, the deflection of each half-frame is directly proportional to the allowable stress only. Thus when the allowable stresses are different on each side, the higher stressed

side will always deflect more, irrespective of which side carries the predominant share of the applied load.

In any particular case, the stresses are specified and the individual deflections are known (e.g., δ_L and δ_R). The ends of the hangars (members 21 and 22) are joined together. Hence the difference in stress between the two hangars is

$$\Delta\sigma_H = \frac{8L}{L'} (\sigma_L - \sigma_R) \quad (33)$$

where L' is the hanger length ($=3600''$) and the subscripts indicate hanger, LHS and RHS, respectively.

Finally, from an inspection of the problem parameters (allowable stress and density), it can be readily decided which half structure should carry the major portion of the applied load. A logical corollary from this is the fact that the hanger on the lightly loaded side will have a minimum area. It follows therefore that the optimum structure will have at least 21 active constraints. The existence or otherwise of the 22nd depends on the values of the parameters.

To complete the optimization, it is assumed that the LHS always has the higher allowable stress. Initially it is taken that the stress/density ratio ensure that the major load is carried on the LHS. Substituting numerical values into Equation 30 the weights of the two halves can be written as

$$\begin{aligned}
 W_L &= \left[\frac{8P_L}{\sigma_L} + 2.554968A^* \right] 360 P_L \\
 W_R &= \left[\frac{8P_R}{\sigma_R} + 2.554968A^* \right] 360 P_R
 \end{aligned} \tag{34}$$

The applied load of the 10^5 lb is the sum of the individual hanger loads, i.e.

$$P_L + P_R = 10^5 \tag{35}$$

$$\text{Hence } W_L = \left[-\frac{8P_R}{\sigma_L} + \frac{8 \times 10^5}{\sigma_L} + 2.554968A^* \right] 360 P_L \tag{36}$$

Since the major load is on LHS, $A_R = A_H^*$ (A_H^* = minimum area of hanger) and

$$\sigma_{R_H} = \frac{P_R}{A_H^*} \tag{37}$$

Because the LHS deflects more than the RHS, the stress in the RH hanger is less than that of the LH hanger.

$$\begin{aligned}
 \text{Then } \sigma_{L_H} &= \sigma_{R_H} - \Delta\sigma_H \\
 &= \frac{P_R}{A^*} - \frac{8L}{L'} (\sigma_L - \sigma_R)
 \end{aligned} \tag{38}$$

The corresponding area A_L is given by

$$A_L = \frac{P_L}{\sigma_{L_H}} = \frac{10^5 - P_R}{\frac{P_R}{A^*} - \frac{8L}{L'} (\sigma_L - \sigma_R)} \tag{39}$$

The total weight is

$$W_{TOT} = W_L + W_R + W_{LH} + W_{RH} \quad (40)$$

While this expression is somewhat complicated algebraically, substitution of numerical values for known parameters simplifies it to the form

$$W_{TOT} = C_1 + C_2 P_R + \frac{C_3}{(P_R - C_4)} \quad (41)$$

where C_1 , C_2 , C_3 & C_4 are numerical coefficients

The weight of the structure is hence a nonlinear function of the single variable P_R . The minimum weight may then be either a free minimum from Equation 41 or may be constrained. Differentiating Equation 41 WRT P_R yields

$$\frac{dW_{TOT}}{dP_R} = C_2 - \frac{C_3}{(P_R - C_4)^2} = 0$$

i.e. $P_R = C_4 + \left(\frac{C_3}{C_2}\right)^{1/2}$ (42)

Since the RH hanger has minimum area, if its stress (Equation 37, computed using P_R given by Equation 42, is greater than its allowable, then the RH hanger is at both minimum area and maximum stress and Equation 42 does not apply. In this case

$$P_R = \sigma_{RH}^* A_H^* \quad (43)$$

and the optimum design has 22 active constraints. With the determination of P_R from either Equation 42 or Equation 43, the optimum design is

completely defined. Both fully- and partially-constrained designs can therefore exist for stress constrained problems. If the stress/density ratio transfers the loading to the RHS, a similar procedure is followed to obtain the optimum designs. Again, depending on the values of the parameters both 21 and 22 active constraint designs may occur.

During the study of this basic problem by various investigators, a number of different combinations of parameters have been used, labeled either Case A or Case B.

To attempt to bring some clarity into the situation, the eight cases given in Table 2 are defined. It is believed that these eight cases encompass the principal problems studied by various investigators. It is left to the reader to correlate the new and old designations.

TABLE 2

22 BAR TRUSS - COMPUTER RESULTS

CASE	LHS			RHS			HANGER		
	σ_{ksi}	ρ	A^*	σ_{ksi}	ρ	A^*	σ_{ksi}	ρ	A^*
1	50	.1	.001	25	.1	.001	5000	.1	.001
2	30	.3	.001	25	.1	.001	5000	.1	.001
3	50	.1	.001	25	.1	.001	500	.1	.001
4	30	.3	.001	25	.1	.001	500	.1	.001
5	50	.1	.01	25	.1	.01	5000	.1	.001
6	30	.3	.01	25	.1	.01	5000	.1	.01
7	50	.1	.01	25	.1	.01	500	.1	.001
8	30	.3	.01	25	.1	.01	500	.1	.001

Using the analytical approach described above, the following optimal solutions have been determined (Table 3).

TABLE 3
22 BAR TRUSS OPTIMAL SOLUTION OF PRINCIPAL CASES

CASE	1	2	3	4	5	6	7	8
WEIGHT	605.096	1202.174	654.04	1232.94	606.752	1312.764	655.70	1236.2
NO. OF CONSTR	21	21	22	22	21	21	22	22

With the determination that stress constrained problems could exhibit less than fully constrained optimal designs it was necessary to revise the previously developed approach to accommodate this situation.

2.7 Constraint Discard and Generalized Newton-Raphson

With the determination that optimal stress limited designs could be less than fully constrained, it became necessary to revise the LPT approach.

From the discussion presented in Section 2.4, it is clear that the operation of a successful search procedure is highly dependent upon some prior knowledge of active constraint population. The LPT will, by definition, always select a full vertex in the linearized design space. If the LPT approach is valid, the optimal population should be a subset of the LPT population. Hence, one approach is to generate a feasible (but non-optimum) full vertex design and then discard unwanted constraints in some rational manner. If a less than fully constrained optimum design is used as an input guess to the LPT routine, a full complement of constraints will be identified, but the Lagrangian multiplier associated with the spurious constraint will be zero. Unfortunately, this does not prove to be of much assistance, since the spurious multiplier generated by the LPT will be non-zero when an off-optimum design is used as input. Also since experience indicates that the values of the multipliers vary by several orders of magnitude in any one problem, it is not practical to use absolute or relative size as a criterion for constraint discard.

After some experimentation with a variety of techniques, which included arbitrary or preferential discard of constraints associated with minimum size members, a new strategy was developed.

In the LPT routine, the variables (Lagrangian multipliers) are restrained to have values greater than or equal to zero. Negative multipliers are not permitted (except those associated with compatibility constraints). Subsequently a new design is created which has active constraints corresponding to the population generated by the LPT routine. For this new design, a new set of Lagrangian multipliers can be generated by linear solution of the optimality criteria equations.

If all the multipliers are positive, the design is optimal. The presence of one or more negative multipliers indicates that non-optimal constraints associated with these negative multipliers leaves a reduced set of potential active constraints for the optimum design.

As discussed in Section 2.4, the use of a Newton-Raphson technique was investigated for the solution of the full set of nonlinear equations consisting of the optimality criteria and all the potential constraints. This approach failed because of the difficulty of identifying meaningful sets of active constraints. If an incomplete set was selected, there was a recurrent problem with singularity.

In the present approach, the number of active constraints is specified a priori. Also the number of constraints is usually a relatively high proportion of the number of variables--thereby seemingly avoiding the problems encountered previously.

The governing nonlinear equations were developed in Section 2.4, along with the Newton linearization.

When the constraint population is known the full set of Newton equations must be reduced to provide the non-singular set to be solved.

If the structure has m elements, each of which has both stress and minimum size constraints and there are n redundancies, the total number of equations available is $(3m + 2n)$. The Newton equations can be arranged in the symmetric array shown in Figure 7.

If a member is known to be at a minimum size constraint, it is not strictly a variable. Hence the corresponding optimality criterion does not apply and can be deleted along with the associated variable, δA_i . Deletion of the variable, δA_i , will result in the elimination of the constraint equation associated with that minimum area. The minimum area constraint equations for unconstrained members do not apply. Hence the complete set of minimum area constraint equations are deleted. Finally, constraint equations and Lagrangian multipliers for non-stress constrained members are not applicable. This reduces the number of applicable equations to the more manageable order $(2n + m + s - a)$ where s is number of stress constraints and a is number of minimum sized members.

When a feasible design has been generated by the LPT section, the corresponding Lagrangian multipliers are determined. Constraints corresponding to any negative multipliers are discarded and the surviving nonlinear equations are solved iteratively using the full Newton-Raphson method.

FIGURE 7. SYMMETRIC ARRAY-NEWTON EQUATIONS

For a structure with four elements and two redundancies the Newton-Raphson array is populated as shown.

	δA_1	δA_2	δA_3	δA_4	δX_1	δX_2	$\delta \mu_1$	$\delta \mu_2$	$\delta \mu_3$	$\delta \mu_4$	δv_1	δv_2	δv_3	δv_4	$\delta \lambda_1$	$\delta \lambda_2$
fA_1	x				x	x					x				x	x
fA_2		x				x		x				x			x	x
fA_3			x			x			x				x		x	x
fA_4				x		x				x				x	x	x
fx_1	x	x	x	x							x	x	x	x	x	x
fx_2	x	x	x	x							x	x	x	x	x	x
fa_1	x															
fa_2		x														
fa_3			x													
fa_4				x												
fs_1	x				x	x										
fs_2		x			x	x										
fs_3			x		x	x										
fs_4				x	x	x										
fe_1	x	x	x	x	x	x										
fe_2	x	x	x	x	x	x										

Condensation:

- (1) All fa_i equations and $\delta \mu_i$ variables are deleted.
- (2) For each element without stress constraint, fs_i equation and δv_i variable are deleted.

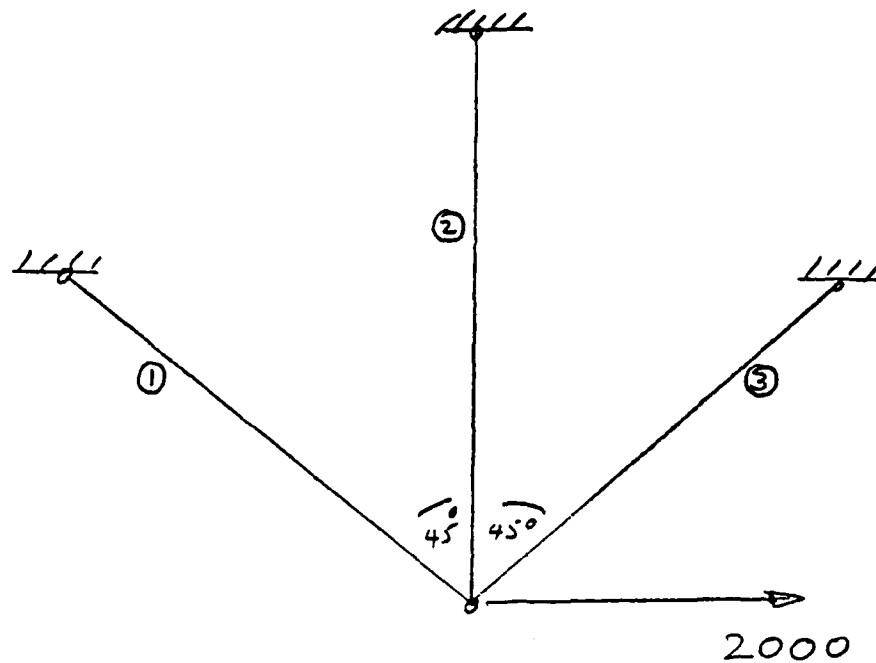
As starting points, the current values of member sizes and redundancies are used. Since the equations are linear in the Lagrangian multipliers, the Newton solution will always generate the same final values for the multipliers, at each step, entirely independent of initially selected guess values. The rate of convergence is therefore unaffected by selection of the starting point for the multipliers. It is controlled by the definition of the constraint population and is also influenced by the initial guesses for the areas.

Also, as is common in many linearized solutions of non-linear problems, the early iterations have to be controlled to prevent instabilities occurring. The usual method used here is the imposition of move-limits on variables. These strategies were used in the program development.

To test the extended computer program, yet another three-bar truss problem was devised. This one is relatively unique in that it has only two active constraints. Details of the problem are given in Figure 8. The program had no difficulty in determining the optimal design.

With the development of the extended program, the 22-bar structures which had contributed so much to the detection of the non-fully constrained problems, could not be handled.

Of the eight problems tested, all except Cases 1 and 3 executed successfully. Table 4 summarizes the numbers of steps required for convergence. In all a single stress-ratio step was used initially to produce a more-or-less feasible design.



$$L = \begin{bmatrix} 1. \\ 1. \\ 1. \end{bmatrix}$$

$$E = \begin{bmatrix} 1. \\ 1. \\ 1. \end{bmatrix}$$

$$\sigma_{max} = \begin{bmatrix} \pm 100. \\ \pm 100. \\ \pm 100. \end{bmatrix}$$

$$A_{min} = \begin{bmatrix} .001 \\ 1.0 \\ .001 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 5.7403 \\ 1000. \\ .1 \end{bmatrix}$$

FIGURE 8 - 3-BAR TRUSS WITH STRESS CONSTRAINTS

As can be seen from Table 4 the results generated correspond exactly to those given in Table 3.

Although no solution could be generated for Cases 1 and 3 using the general starting point, the program did recognize the optimal designs when supplied as input. Table 5 provides details.

TABLE 4
22 BAR TRUSS - ITERATION HISTORY

CASE NO.	STRESS RATIO ITERATION	NO. OF L-P ITERATIONS	NO. OF N-R ITERATIONS	FINAL WEIGHT	NO. OF ACTIVE CONSTRAINTS	REMARKS
1	1	2				L-P Stages Diverged
2	1	2	7	1202.17	21	
3	1	5				L-P Stages Diverged
4	1	2		1232.9	22	Final Solution is linear
5	1	2	4	606.75	21	Fastest Convergence
6	1	2	11	1312.76	21	Slow N-R Convergence
7	1	2		655.70	22	
8	1	2		1236.2	22	Final Solution is linear

TABLE 5
22 BAR TRUSS - TEST CASE RESULTS USING OPTIMAL INPUT

CASE NO.	STRESS RATIO	NO. OF L-P ITERATIONS	NO. OF N-R ITERATIONS	FINAL WEIGHT	NO. OF ACTIVE CONSTRAINTS	REMARKS
1	0	1	4	605.094	21	Optimal Solution Input
3	0	1		654.04	22	

Before assessing the reasons for the difficulties with Cases 1 and 3, it is appropriate to evaluate some of the computational requirements of the new procedure.

The 22-bar truss has 5 redundancies. Hence, each analysis requires solution of 5 equations. The stress-ratio requires one such analysis.

The LPT stage extracts the optimal solution from a set of $(22+5) = 27$ equations involving $(2 \times 22 + 5) = 49$ unknowns. Typically, the solution requires 40-50 row/column interchanges. A double precision Bell LPT routine is used. The double precision was introduced because of an early impression of conditioning problems. There is no evidence that double precision is necessary for the cases actually solved. The reduced N-R used between LPT stages requires the iterative solution of $(5 + \text{NIQ})$ equations, where NIQ is the number of doubly defined elements. When the solution is linear, 5 equations are solved once.

In the major N-R, the number of symmetric equations solved was only 31, since 10 stress constraints and 11 minimum areas were defined for the optimal population. The generation of all the terms in the array solved is simple, direct and requires no auxiliary analyses to generate approximate derivatives.

Turning to the two cases on which the program failed, there appears at first glance to be no obvious reason for the inability of the LPT routines to generate feasible constraint distributions. It is true that these

cases represent the most radical variations in allowable stresses coupled with extremely small minimum areas. Examination of the values of the Lagrangian multipliers does indicate a potential conditioning problem. The sum of the Lagrangian multipliers is the merit condition for the LPT. In Case 1, the multipliers range in value from 140.31 down to .00276 - which immediately suggests a potential problem. In the successful Case 2, the corresponding range is 283.77 to .00276 - and no difficulty was encountered. It is, nevertheless, entirely possible that a conditioning problem does exist in the LPT stage. Switching to double precision in the LPT routine may be only half the solution. It may be necessary, for problems as extreme as those considered herein, to formulate the terms in the LPT in double precision.

Finally, while comparisons are odious, it is relevant to compare the present solutions for 22-bar trusses with a few available from alternate sources.

The 22-bar was discussed extensively in Reference 9. Two cases were presented. From the results, these are believed to correspond to Cases 1 and 2 of the present report. Case 1 yielded a final weight of 606 lb. and Case 2 a weight of 1203 lb. Both required a large number of iterations to converge. For Case 2, an even lower weight of 1188 lb. is quoted, but it is not clear as to whether this represents a truly feasible design.

The team of Bartholemew and Morris at RAE has developed the SCICON structural optimization system⁽¹⁴⁾ and applied it to the 22-bar problem⁽¹³⁾. For Case 2 a weight of 1204.38 lb. was generated in 15 iterations. This was not a converged design, as was shown by the duality gap between the above primal weight and a dual weight of 1200.66 lb. The optimization may have been terminated either due to a limit on number of iterations or because there is practically no weight change between the last two iterations. The final design presented is close to the optimum but has a significant number of constraints (particularly minimum areas) which are not considered active.

That this situation should occur is puzzling since a comparison of the two methods shows very few real differences. In each case, an LPT step (or steps) is used to determine a constraint population followed by a N-R search. The SCICON program allows both accumulation and discard of constraints. This strategy was not found necessary for the present problems but is used in later versions of the program. The statement is made by RAE, that the number of active constraints is typically much smaller than the number of variables - a fact not really borne out in the current examples. The RAE solution requires the solution of an $m \times m$ matrix, where m is the number of active constraints. The $m \times m$ matrix is itself a triple product $(G H^{-1} G^T)$ where the H is a diagonal $n \times n$ matrix with n as the number of variables. Finally, it should be noted that G is a constraint derivative matrix. As is well known, if the constraints are stresses, treating the internal forces as invariants can lead to gross errors. To include the

internal force redistribution effects in computing the derivatives either requires a finite difference method or the $\frac{\partial q}{\partial A} = -S K^{-1} \frac{\partial K}{\partial A} \Delta$ formulation. Both of these methods are computationally expensive. Due to the known extreme nonlinearity of the problem, it is entirely possible that the convergence characteristics are strongly influenced by the accuracy of the constraint derivative matrix G.

With the force method, the constraint derivative matrix is always exact and is generated with a minimum of auxiliary calculation.

2.8 Displacement Constraints

The next stage is the formal incorporation of displacement constraints. In principle, this does not raise any new problems, since displacement constraints have already been treated in the form of the compatibility conditions.

The only obvious feature of external displacement constraints are that they (a) are associated with non-zero constraint values and (b) are inequality constraints which need not be satisfied in the equality sense. Compatibility conditions are zero-valued and must be satisfied as equalities throughout.

The governing Lagrangian is modified as shown in Figure 9. Figure 9 also gives the modified forms of the derivatives (optimality criteria and constraints). In the modified equations, one new matrix parameter is introduced (B_D). This is the virtual force in each member of the structure arising from the (virtual) unit loads imposed on the structure to calculate the displacements. Strictly from the definition of the virtual load method B_D need only be a system in static equilibrium with the virtual load. In the present context it has been shown⁽²⁾, that the B_D system must be the actual internal force distribution arising from the virtual force.

Because of the additional displacement related terms, the linearized Newton-Raphson equations must also be expanded as indicated in Figure 10. Naturally only active displacement constraints will be retained in the N-R equations to be solved. The total number of equations is given by $(2n + m + s - a + d)$ where d is the number of active displacement constraints.

Additional terms required for displacement constraints

Lagrangian

$$L = \text{---} + \sum_{\ell=1}^q \gamma_{\ell} \left[\frac{1}{c_{\ell}} \sum_{i=1}^m (B_0^i + \sum_{k=1}^p B_1^{ik} x_k) \bar{f}_i B_D^{il} - 1 \right]$$

Optimality Criteria

$$\frac{\partial L}{\partial A_i} = 0 = \text{---} - \sum_{\ell=1}^q \gamma_{\ell} \frac{1}{c_{\ell} A_i^2} \left[(B_0^i + \sum_{k=1}^p B_1^{ik} x_k) \bar{f}_i B_D^{il} \right]$$

$$\frac{\partial L}{\partial x_j} = 0 = \text{---} + \sum_{\ell=1}^q \gamma_{\ell} \frac{1}{c_{\ell}} \sum_{i=1}^m \frac{B_1^{ij} B_D^{il} \bar{f}_i}{A_i}$$

Displacement Constraint

$$\frac{\partial L}{\partial \gamma_{\ell}} = \frac{1}{c_{\ell}} \sum_{i=1}^m (B_0^i + \sum_{k=1}^p B_1^{ik} x_k) \bar{f}_i B_D^{il} - 1 = f_{D_{\ell}}$$

FIGURE 9 - MODIFIED LAGRANGIAN TO INCLUDE
DISPLACEMENT CONSTRAINTS

Optimality Criteria

$$\frac{\partial f_{A_i}}{\partial A_i} \delta A_i + \sum_{j=1}^p \frac{\partial f_{A_i}}{\partial X_j} \delta X_j + \frac{\partial f_{A_i}}{\partial \mu_i} \delta \mu_i + \frac{\partial f_{A_i}}{\partial v_i} \delta v_i + \sum_{j=1}^p \frac{\partial f_{A_i}}{\partial X_j} \delta X_j + \sum_{\ell=1}^q \frac{\partial f_{A_i}}{\partial \tau_\ell} \delta \tau_\ell = -f_{A_i}$$

$$\sum_{i=1}^m \frac{\partial f_{X_j}}{\partial A_i} \delta A_i + \sum_{i=1}^m \frac{\partial f_{X_j}}{\partial v_i} \delta v_i + \sum_{k=1}^p \frac{\partial f_{X_j}}{\partial \lambda_k} \delta \lambda_k + \sum_{\ell=1}^q \frac{\partial f_{X_j}}{\partial \tau_\ell} \delta \tau_\ell = -f_{X_j}$$

Constraint Conditions

$$\frac{\partial f_{A_i}}{\partial A_i} \delta A_i = -f_{A_i} ; \quad \frac{\partial f_{S_i}}{\partial A_i} \delta A_i + \sum_{j=1}^p \frac{\partial f_{S_i}}{\partial X_j} \delta X_j = -f_{S_i}$$

$$\sum_{i=1}^m \frac{\partial f_{\epsilon_j}}{\partial A_i} \delta A_i + \sum_{k=1}^p \frac{\partial f_{\epsilon_j}}{\partial X_k} \delta X_k = -f_{\epsilon_j} ; \quad \sum_{i=1}^m \frac{\partial f_{D_\ell}}{\partial A_i} \delta A_i + \sum_{k=1}^p \frac{\partial f_{D_\ell}}{\partial X_k} \delta X_k = -f_{D_\ell}$$

-70-

Additional derivatives required are $\frac{\partial f_{A_i}}{\partial \tau_\ell} (= \frac{\partial f_{D_\ell}}{\partial A_i})$ and $\frac{\partial f_{X_j}}{\partial \tau_\ell} (= \frac{\partial f_{D_\ell}}{\partial X_k})$. Other derivatives must be modified to include terms for the displacement constraints. The form of all additional equations and terms corresponds to that used for compatibility.

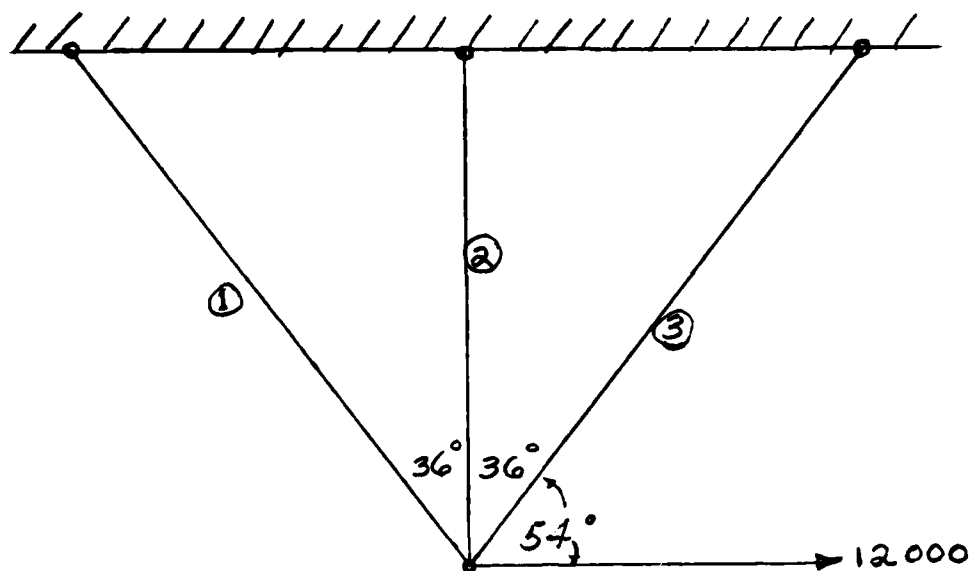
FIGURE 10 - MODIFIED NEWTON EQUATIONS TO INCLUDE DISPLACEMENT CONSTRAINTS

The additional term, d , does not necessarily mean an increase in the order of the equations since the total number of active constraints can still never exceed the number of variables. The upper limit on the number of equations is $(2n + 2m - 1)$ which would only occur when no minimum size constraints existed.

Because of the similarity between the compatibility and displacement constraints, modification of the computer program was relatively straightforward. The B_D matrices necessitated the introduction of additional loading cases, but this concept had been used in the earlier OPTIM programs.

The new program was tested on small scale 3-bar truss problems as usual. Figure 11 provides details. A few minor developmental problems were encountered. More limits were introduced on the early iterations to prevent initial transients producing unstable situations. Also the ability to acquire and discard constraints was also incorporated.

After each iteration, following the initial transients--estimated as dying out in 2 or 3 iterations, the generated design was compared with all constraints. If any additional constraints had become activated, they were duly recognized and considered in subsequent steps. If the number of recognized constraints equals or exceeds the number of variables, the program would then return to the LPT mode to select a new design. Discard of constraints was based on the same criterion as used initially--i.e., non-positive Lagrangian multipliers.



$$L = \begin{bmatrix} 125. \\ 100. \\ 125. \end{bmatrix}$$

$$\rho = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

$$A_{MIN} = \begin{bmatrix} 0.01 \\ 0.01 \\ 0.01 \end{bmatrix}$$

$$\sigma_{MAX} = \begin{bmatrix} \pm 50. \\ \pm 50. \\ \pm 50. \end{bmatrix}$$

$$E = \begin{bmatrix} 10000. \\ 10000. \\ 10000. \end{bmatrix}$$

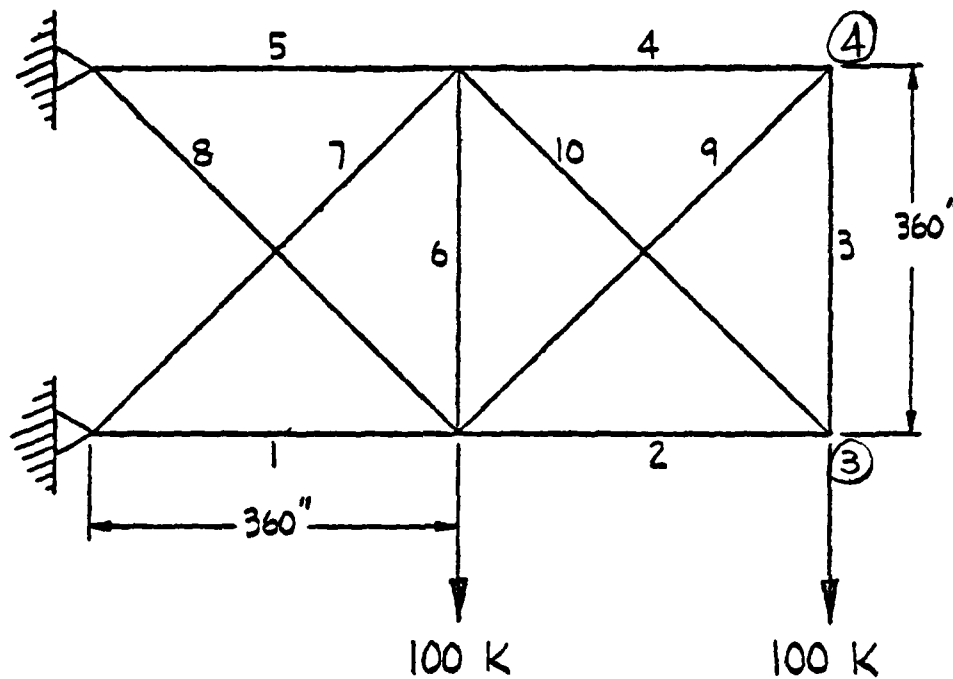
$$\delta_{x,y} = \pm 0.05$$

FIGURE 11 - 3-BAR TRUSS, WITH DISPLACEMENT CONSTRAINTS

The next problem tested is another variant of the 10-bar truss structure. This problem used in Reference 1, proved to be a rather difficult one to solve and no exact optimum solution has been published. The problem is shown in Figure 12. The major difficulty arises from the specification of constraints on the vertical displacements at both the lower and upper nodes at the free end. If it is assumed, say, that the lower node will deflect more than the upper and the structure is optimized using FSD and the envelope method with the single displacement constraint, the upper node in the optimized system experiences an excessive displacement. Using the upper displacement as the constrained value reverses the situation and the lower displacement violates at the new optimum. Using both upper and lower displacements in the envelope method produces an unsatisfactory situation in which no converged solution is obtained. In the course of the iterations, which tend to become unstable at one stage, a very low weight design is generated. From examination of this design, it does appear to be near optimum. Unfortunately it has not been possible to determine exactly what the true optimum is and which constraints are active.

Using the current version of the program the above problem was tried. A number of difficulties were encountered.

In the first place, it proved to be impossible to generate a feasible full vertex design, based upon any population predicted by an LPT solution. Although a number of iterations were permitted, resulting in a number of



	A_{MIN} (IN ²)	σ_{MAX} (PSI)	δ_{MAX} (IN)
MEM 1-10	0.1	25,000	± 2.0 AT PTS. 3 & 4

FIGURE 12 - 10-BAR TRUSS WITH
DISPLACEMENT CONSTRAINTS

different LPT populations, it was never possible to converge on any full-vertex designs using the Newton-Raphson technique. This situation had been encountered previously, even for stress limited problems. The solution adopted at that time was to generate a fully stressed design (or a close approximation thereto) using a limited number of stress-ratio iterations. For this design, the Lagrangian multipliers were found and after any constraint discard, the full Newton-Raphson search was used. In the presence of potentially active displacement constraints, the FSD alone may result in a violated design. From expediency, it was decided to use this FSD suitably scaled to ensure satisfaction of the single dominant displacement constraint as the starting point for the Newton search. Consideration was also given to the use of an envelope method to generate a starting point, either using a single or multiple displacement constraints. This approach was not implemented in the present program for a number of reasons. Firstly, the envelope method is computationally more complex than the stress-ratio FSD. Secondly, from the previous history with this particular problem, use of either of the envelope methods could be unreliable--a single constraint always failed and double constraints would not converge.

Using this scaled FSD as a starting point, the Newton-Raphson stage was entered. For a limited number of steps, convergence appeared to be satisfactory--moving in the direction of the assumed optimal design. But after additional constraints were captured, according to the built-in logic, the direction of convergence altered and rapidly became unstable.

Table 6 provides details.

10-BAR TRUSS WITH DISPLACEMENT CONSTRAINTS - COMPUTER RESULTS

INCLUDE MAIN
 INCLUDE CM1
 INCLUDE STW2
 INCLUDE CSUBS
 INCLUDE DISP
 END RUN

PAUSE ENTER IRD, ISR, ISCALE, IPRINT
 0.1,1,-3

STRESS RATIO AREAS

1	0.30373E 02	2	0.88870E 01	3	0.59555E 01	4	0.59555E 01	5	0.28997E 02
6	0.52675E 01	7	0.20017E 02	8	0.21953E 02	9	0.84222E 01	10	0.12568E 02

X 1 -0.13894E 02 2 0.30352E 02

27 1

MEM 1 2 3 4 5 6 7 8 9 10

AREA 0 0 1 0 0 1 0 0 0 0

STR -1 1 0 0 1 0 -1 1 1 0

DISPL 1 2

ACTIVE 1 1

N-R AREAS

1	0.80000E 01	2	0.40000E 01	3	0.10000E 00	4	0.10000E 00	5	0.80000E 01
6	0.10000E 00	7	0.56548E 01	8	0.56548E 01	9	0.10000E 00	10	0.10000E 00

X VALUES 1 -0.48369E-03 2 0.43948E-04

N-R DID NOT CONVERGE. REENTER L-P

27 1

MEM 1 2 3 4 5 6 7 8 9 10

AREA 0 0 0 0 0 1 0 0 0 1

STR -1 -1 0 0 0 -1 1 1 -1 -1

DISPL 1 2
 ACTIVE 1 1
 N-R AREAS
 1 0.11904E 02 2 0.10000E 00 3 0.35122E 06 4 0.10000E 00 5 0.14967E 01
 6 0.16486E 00 7 0.13315E 00 8 0.11180E 02 9 0.57568E 01 10 0.10000E 00
 X VALUES 1 -0.97646E 02 2 0.10177E 03
 STRESS RATIO AREAS
 1 0.29026E 02 2 0.14176E 02 3 0.36001E 00 4 0.36001E 00 5 0.28575E 02
 6 0.36001E 00 7 0.20046E 02 8 0.20684E 02 9 0.36001E 00 10 0.20048E 02
 X 1 -0.15550E 01 2 0.15536E 01

SINGLE DISPLACEMENT CONSTRAINT DOMINATES

ITER NO. 2

SUM MEM MBT = 0.57355E 04 SUM LAM = 0.57355E 04
 N-R AREAS
 1 0.21374E 02 2 0.14281E 02 3 0.44131E 00 4 0.44131E 00 5 0.30355E 02
 6 0.21601E 00 7 0.21989E 02 8 0.12410E 02 9 0.50402E 00 10 0.20197E 02
 0.17070E 01 0.87457E 00
 0.5221102E 04 0.4374711E 04
 9 0 1 0
 N-R AREAS
 1 0.20732E 02 2 0.15925E 02 3 0.26478E 00 4 0.26478E 00 5 0.30621E 02
 6 0.15470E 00 7 0.22099E 02 8 0.74462E 01 9 0.30241E 00 10 0.22521E 02
 0.43817E 01 -0.40262E 00
 0.5112699E 04 0.4502434E 04
 9 0 1 0
 N-R AREAS
 1 0.22290E 02 2 0.15085E 02 3 0.15887E 00 4 0.15887E 00 5 0.30860E 02
 6 0.10607E 00 7 0.21748E 02 8 0.54855E 01 9 0.18553E 00 10 0.21334E 02
 0.46337E 01 -0.6335E 00
 0.4953781E 04 0.4788637E 04
 9 0 1 0

TABLE 6 (CONT'D)

N-R AREAS
 1 0.20604E 02 2 0.15652E 02 3 0.10000E 00 4 0.10000E 00 5 0.31637E 02
 6 0.14849E 00 7 0.23173E 02 8 0.32913E 01 9 0.11132E 00 10 0.22135E 02
 0.10773E 02 -0.10200E 01
 0.4936584E 04 0.4581691E 04
 4 2 2 2
 N-R AREAS
 1 0.22037E 02 2 0.15565E 02 3 0.10000E 00 4 0.10000E 00 5 0.30110E 02
 6 0.18551E 00 7 0.21093E 02 8 0.46078E 01 9 0.15585E 00 10 0.22012E 02
 0.86500E 01 -0.62550E 00
 0.4888543E 04 0.5920930E 04
 4 2 2 2
 N-R AREAS
 1 0.22552E 02 2 0.15364E 02 3 0.10000E 00 4 0.10000E 00 5 0.30453E 02
 6 0.20382E 00 7 0.21233E 02 8 0.64510E 01 9 0.13928E 00 10 0.21728E 02
 0.61123E 01 -0.23041E 00
 0.4998473E 04 0.5225891E 04
 4 2 2 2
 N-R AREAS
 1 0.22810E 02 2 0.14992E 02 3 0.10000E 00 4 0.10000E 00 5 0.30930E 02
 6 0.17419E 00 7 0.21601E 02 8 0.65412E 01 9 0.10000E 00 10 0.21202E 02
 0.52540E 01 -0.18723E 00
 0.5005059E 04 0.3111493E 04
 3 2 2 3
 N-R AREAS
 1 0.28306E 02 2 0.89953E 01 3 0.10000E 00 4 0.10000E 00 5 0.39001E 02
 6 0.14835E 00 7 0.27401E 02 8 0.71342E 01 9 0.10000E 00 10 0.12721E 02
 0.41969E 01 0.91389E-01
 0.5170379E 04 0.1882996E 04
 3 2 2 3
 N-R AREAS
 1 0.31648E 02 2 0.96492E 01 3 0.10000E 00 4 0.10000E 00 5 0.43846E 02
 6 0.11804E 00 7 0.30818E 02 8 0.70263E 01 9 0.10000E 00 10 0.13646E 02
 0.36351E 01 0.51533E-02
 0.5703102E 04 0.1138443E 04
 3 2 2 3
 N-R AREAS
 1 0.28042E 02 2 0.93600E 01 3 0.24000E 00 4 0.10000E 00 5 0.41884E 02
 6 0.10000E 00 7 0.29597E 02 8 0.59015E 01 9 0.10000E 00 10 0.13236E 02
 0.39652E 01 -0.14401E 00
 0.5356340E 04 -0.5979176E 04
 2 2 2 4
 0 STOP 6200

TABLE 6 (CONT'D)

After considerable numerical experimentation the problem appeared to reside in the accumulation of undesirable constraints during the search, with no satisfactory indications which should be rejected. Thus the problem again appeared to devolve on the question of constraint identification.

3.0 PROGRAM DEVELOPMENT AND ORGANIZATION

The Force Method Optimization (FMO) program was developed and tested using the Bell RAX system. The convenience and ready access of the RAX system make it attractive for use in structural and other program development, although the RAX core storage capacity has limited the FMO test problem size to no more than that required for solution of the 22-bar truss. Very recently, a Virtual Storage system (VSPC) has been installed at Bell which considerably increases the RAX storage capacity, but was not yet in place during the program development.

A structure cutter was also developed and tested as part of the FMO program development, but was not integrated in the program since it would have further reduced the problem size which could be analyzed on the RAX system. The essential or basic function of the structure cutter in this application, i.e., provide values of the b_0 and b_1 matrices, was performed by hand calculation and input as basic data for most of the test problems analyzed during program development.

Much discussion of how the FMO program evolved to its present form is sprinkled throughout the Technical Discussion, Section 2.0, and the reader is referred to this discussion for clarification of specific points regarding why a particular action or step was required. Also, since the FMO RAX program is reasonably straightforward in programming style, it does not appear to be necessary to describe in detail the programming steps which were followed to implement the stages of program development.

Hence, the principal item to be discussed in this section is the organization of the FMO RAX program. A flow diagram of the RAX program is presented in Figure 13. Referring to the flow diagram, material properties and other data pertaining to the structure to be analyzed are input to the program as basic data. The freefield format is used, as shown in Figure 14, for the 3-bar truss of Figure 11. The data to be entered consists of the number of members, redundancies, and displacement constraints in the first line, followed by the densities, elastic moduli, and member lengths in the second, third, and fourth lines. The selected values of minimum member area and the upper and lower stress limits are entered in the fifth, sixth, and seventh lines, followed by inputs for the displacement limits, and the calculated values of b_0 and b_1 . This completes the basic data entries.

A starting point design (guess vectors) can be input after the basic data entry, or the default option exercised, which automatically selects minimum areas as the initial point design. Based on experience, a closer approximation to the optimal system may be obtained using a stress-ratio design. Hence, a stress-ratio cycle is available to generate a "better" initial point design, as an option.

The starting or initial point design is applied to the LPT routine which generates a potential constraint population by solving a set of non-square linear equations in $(m + p)$ unknowns corresponding to the optimality criteria. The LPT routine was not developed during this program. It is a mathematical routine obtained from the Bell library of computational routines. It also

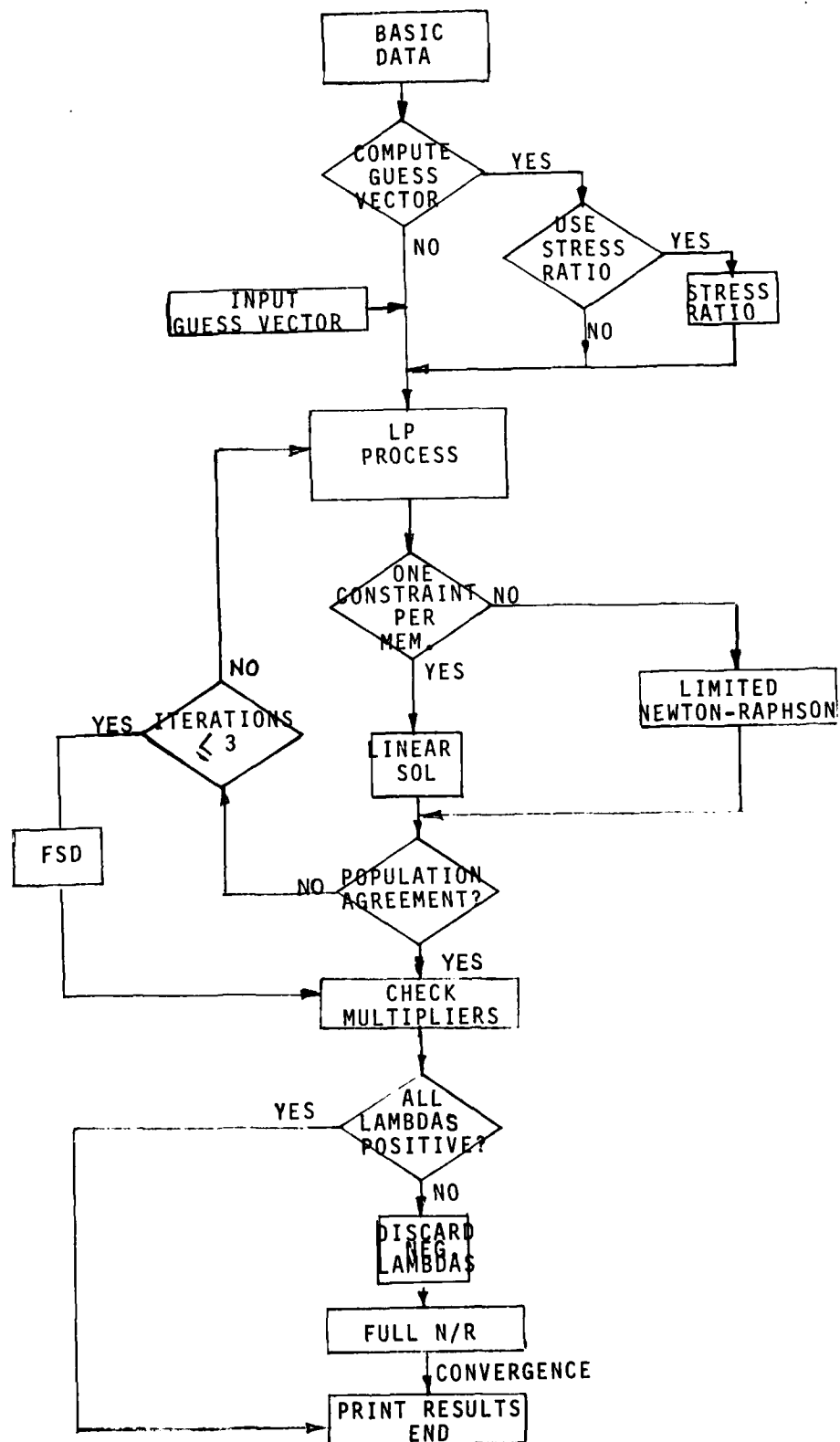


FIGURE 13. FMO COMPUTER PROGRAM - FLOW DIAGRAM

Line 1	3, 1, 2	No. of Members, Redundancies, Displ. Const.
2	.1, .1, .1	Density, lbs/cu. ft.
3	10000., 10000., 10000.	Elastic Moduli, psi
4	125., 100., 125.	Length, inches
5	.01, .01, .01	Minimum Area, sq. in.
7	50., 50., 50.	Upper Stress Limit, psi
8	-50., -50., -50.	Lower Stress Limit, psi
9	1., .1, .1	Displacement Limit, in.
10	20., -7., 0.	b_0 matrix
11	1.66667, -1.3333, 0.	b_1
12	0., 1., 0.	matrix
13	1., -1.6, 1.	

FIGURE 14 - BASIC DATA - INPUT FORMAT

generates values for the non-zero Lagrangian multipliers. If the population shows that each member is constrained, i.e., either a minimum area or a maximum stress is present in each member, the solution of linear equations leads to the required internal force distribution which satisfies the prescribed FSD conditions without iteration. If the constraint population is other than a FSD type, i.e., a member may be defined as both fully stressed and at minimum area, this will result in another member being completely unrestrained. The set of compatibility equations must be expanded by additional relationships. The number of variables increases to include redundancies and undefined areas and the equations become nonlinear. A Newton-Raphson routine is used to generate a design corresponding to the specified constraints. The constraint population which results from this design is checked for correspondence with the original LPT population output (optimality criteria). If this population check fails, the constrained design is iterated (no more than 3 times) through the LPT routine until population agreement is obtained. A FSD is generated if population agreement cannot be obtained after three iterations, and the program passes out of the LPT loop.

As noted in the flow diagram, a check of the Lagrangian multipliers is performed at this point. If, as discussed in Sections 2.5 and 2.6, all multipliers associated with non-zero type constraints have positive values, the design is considered to be optimal, and the results are printed, ending the program operation. The alternate condition, i.e., one or more negative multipliers indicates that non-optimal constraints are present. Discarding

the constraints associated with these negative multipliers leaves a reduced set of potential active constraints for the optimum design. The resulting nonlinear equations are solved iteratively using the Newton-Raphson routine until convergence is obtained, indicating an optimum design has been reached. Results are printed, ending program operation.

4.0 CONCLUSIONS

The new approach to structural optimization via the use of the force method of analysis and a linear programming stage has achieved a major degree of success. Some of the classic optimization test problems have been reduced to almost trivial solutions. The 10-bar truss problem with non-uniform stresses is solved without iteration. On the larger 22-bar trusses, solutions have been achieved in the majority of cases, which are confirmed as being optimal by comparison with analytically generated results. The results provide a clear indication of the validity of the approach for the rigorous incorporation of stress constraints. Because of the relatively rapid convergences achieved, the iterative computational effort appears to be significantly lower than that required by alternate approaches. The key to the generation of the optimal design lies in the ability to predict or identify the active constraints. The linear programming stage incorporated in the developed approach does an excellent job of selecting dominant constraints through local linearization of the non-linear domain, but, as in any non-linear situation, does require either a good starting point and/or a well conditioned problem to solve.

In the study of the 22-bar problems, no solutions could be generated using the standard program, although the correctness of the optima were demonstrated by the program. In these cases, one of which was a full vertex design, the problem was felt to be attributable to conditioning. This could not be entirely proven, since some of the other 22-bar cases which

were successfully solved might have appeared to be more poorly conditioned. The conditioning problem here was taken to be associated with the very large spreads in magnitude between the largest and smallest Lagrangian multipliers. The spread is 10^5 and since the merit function is the sum of the multipliers, there is potential for round-off error to obscure the situation and lead to erroneous definition of active constraints.

Naturally, the 22-bar problem is an unrealistic one selected to exhibit certain characteristics on a relatively small scale and to be a test for optimization techniques. It is not clear whether the problem is really pathological - especially in its ranges of values of multipliers - or if it is truly characteristic of larger scale problems. Some consideration has been given to transformation of the problem to try and eliminate the size disparity in numbers which are linearly combined. Simple scaling alone will not suffice, but no suitable transformation could be evolved.

The presence of very small, but non-zero, values for Lagrangian multipliers associated with real (active) constraints presents a hazard in the Newton-Raphson search procedure which allows for constraint acquisition and discard. The question of when a multiplier becomes zero and the corresponding constraint should be discarded is made extremely difficult, since such small values are readily masked by round-off error in large scale problems.

With regard to displacement constraints, the fundamental concepts and approach seem to have been validated. Some small scale problems were

successfully solved. Difficulties with constraint detection on larger problems appear to remain. In purely stress and minimum size problems, the number of active constraints equals or approaches the number of variables. The population defined by the LPT stage is therefore a reliable indication of the final active constraints. In the presence of displacement constraints, this no longer holds true. The number of active constraints is frequently very much reduced and the LPT is much less reliable in predicting population.

To overcome this situation, the expedient solution of selecting a single dominant displacement constraint was adopted to generate a starting design for the Newton search. This did not perform satisfactorily and placed too much reliance on the constraint acquisition/discard algorithms. It is clear that the method of selecting a starting design for the Newton search must be revised, but in what direction requires some further consideration.

Although difficulties were encountered in the incorporation of displacement constraints per se, it is believed that these are operational problems of technique and strategy rather than of concept.

In all optimization research, there has always been a great deal of concern for computational efforts involved in the various approaches. Both the numbers of iterations and the cost per iteration are major contributing factors. In this force method work, the costs appear to compare most favorably with other methods. The force method, generally,

requires the solution of a smaller array of equations than the displacement method for a given structure (redundancies versus displacements). The need for a structure cutter in the force method has a major cost impact in a single-shot analysis but this effect is reduced considerably in iterative situations. The LPT stage is equivalent to the inversion of an array of order of the number of variables. The full Newton-Raphson search does require consideration of a larger set of equations - iteratively. For the 22-bar trusses 31 equations are required. Although this array is large, the computation of the individual terms is very straightforward. No auxiliary analyses are required to generate approximations to gradients of constraints. The absence of approximations throughout is considered to be a major factor in the good convergence characteristics observed. The size of the Newton array may be a problem in larger systems. The matrix is symmetric and sparse, so that the possibility of using efficient solution techniques exists.

In view of the successful solutions demonstrated, it is recommended that development of this force method approach be continued. Clearly, the problems encountered will have to be overcome. Further, study of the conditioning problem in the LPT stage may be solvable either through some transformation or use of a more sophisticated LPT routine.

For displacement constraints, a method of selecting a better starting point for the N. R. search is required when the LPT is unsatisfactory. Also the strategies for acquisition and discard of constraints during the N. R. stage may require further treatment.

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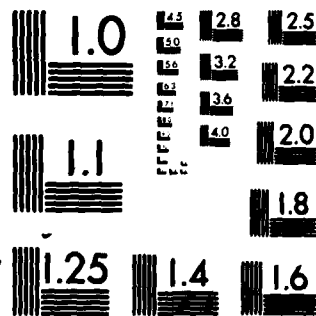
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Finally, the search for alternate methods of handling the entire problem of constraint identification should be pursued. The present work is felt to have been a positive step in the direction of generating an operational optimization program and provides a firm basis for future work in this complex but rewarding technology.

5.0 REFERENCES

1. R. A. Gellatly and L. Berke, "Optimal Structural Design," AFFDL-TR-70-165, Feb. 1971.
2. R. A. Gellatly, D. M. Dupree and L. Berke, "OPTIM II, A MAGIC - Compatible Large Scale Automated Minimum Weight Design Program," AFFDL-TR-74-97, 1974.
3. B. J. Dale, J. Padlog, and R. D. Thom, "OPTIM III: A NASTRAN Compatible Large Scale Automated Minimum Weight Design Program," Bell Report No. D2530-933022, Bell Aerospace Textron, September 1979.
4. W. J. Dwyer,, R. K. Emerton and L. U. Ojalvo, "An Automated Procedure for the Optimization of Practical Aerospace Structures," AFFDL-TR-70-118.
5. N. S. Khot, L. Berke and V. B. Venkayya , "Minimum Weight Design of Structures by the Optimality Criterion and Projection Method," 20th Structures and Structural Dynamics Conference, St. Louis, Missouri, April 1979.
6. G. Thierau and A. Topcu, "Structural Optimization Using the Force Method," World Congress on Finite Element Methods in Structural Mechanics, England, Oct. 1975.
7. W. Lipp and G. Thierauf , "Die Bedeutung des Kraft - und Weggrossenverfahrens fur die Optimierung von Tragwerken nach der Lagrange'schen Multiplikatorenmethode," Tenth Congress, IABSE, Tokyo, Sept. 1976.
8. R. A. Gellatly and L. Berke, "A Preliminary Study of a New Approach to the Optimization of Strength Limited Structures," AFFDL-TM-75-162-FBR, Sept. 1975.
9. L. Berke and N. S. Khot, "A Simple Virtual Strain Energy Method to Fully Stress Design Structures with Dissimilar Stress Allowables and Material Properties," AFFDL-TM-77-28-FBR, December 1977.

10. J. Pichard, "FORMAT II - Second Version of Fortran Matrix Abstraction Technique: Engineering User Report," Volume I, AFFDL-TR-66-207, Air Force Flight Dynamics Laboratory, WPAFB, Ohio, September, 1965.
11. P. Bartholomew and A. J. Morris, "A Unified Approach to Fully-Stressed Design," Engineering Optimization, 1976, Vol. 2, pp 3-15.
12. L. Berke and N. S. Khot, "Use of Optimality Criteria Methods for Large Scale System" AGARD Lecture Services No. 70 on Structural Optimization, AGARD-LS-70, 1974.
13. Private communication from P. Bartholomew, 1978.
14. W. Hay and N. A. Wheble, "SCICON, Description of a Structural Optimisation Package, Phase I - Prototype," Scientific Control Systems Limited, August 1977.
15. J. S. Arora and A. K. Govil, "An Efficient Method for Optimal Structural Design by Substructuring," Comput. Structures 7(4), 507-515 (1977).